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Teaching the Concept of Unit in Measurement Interpretation of Rational Numbers

H. Bahadir Yanik*   Brandon Helding**   Alfinio Flores***

ABSTRACT: This study investigated middle school students understanding of unit and unitization concepts in measurement interpretations of rational numbers using the number line as a tool. Fifty-six seventh-grade students were pretested and five consecutive whole-class teaching experiments were developed and administered based on pretest results. Five students were later chosen for semi-structured clinical interviews, based on their conceptions of unit and unitization. Students' reasoning was induced from the analysis of pre- and post-tests, observations of classroom teaching episodes, videotapes of interviews, and transcriptions and photographs of student artifacts. Results suggested that unit identification created difficulty for students in locating rational numbers on number lines.
Key Words: Rational Numbers, Unit, Unitization, Measurement

SUMMARY

Purpose and significance: The purpose of this study was to investigate middle school students' difficulties understanding the "unit" associated with measurement interpretations of rational numbers using number lines. We showed what kind of "unit" understanding is necessary for a measurement interpretation of rational numbers and how classroom instruction can support the development of such understanding, especially building on students' part-whole interpretations of unit.

Methods: A total of fifty-six seventh grade students from an urban K-8 school located in southwestern United States participated in this study. Students from three mathematics classes were pretested, based on the existing literature concerning length measurement and number lines, to determine their initial understanding. Based on the data gathered, a teaching experiment was conducted to see how students' understanding developed over a three-week period. Data was gathered through classroom observations, student artifacts, and interviews. Students demonstrated several misconceptions when using number lines as a tool for measurement.

Results: Results indicated that students had problems locating improper fractions on the number line. For instance, fraction notation was read inverted to make improper fractions into proper fractions, resulting in a wrong answer. Furthermore, students misinterpreted the whole number line as the unit rather than a connected, continuous composition of units. The majority of students mislocated proper fractions on a number line from 0 to 5. For instance, when locating ¾ on the number line (ranging from 0 to 5), these students partitioned the number line into four equal pieces and then marked the 3rd point (partition) to the right of zero.

Discussion and Conclusion: Students' difficulties in applying rational numbers as measures on number lines revealed that the unit and unitization concepts do not develop naturally. When using an abstract tool such a number line, students did not easily see fractions as measures of distance. Using concrete materials as a unit and focusing on the iteration of that unit help students to develop a measurement "sense". Moreover, transition from concrete representations to more abstract representations (e.g. number line) is critical to fully understanding rational numbers as measures. Several instructional strategies help students see fractions as measures. Effective instruction should focus on the important concepts within measurement such as unit, unitization, and the iteration of length units; and connect key mathematical content to students' previous knowledge and experience. Also, group activities need to actively engage children in meaningful representations and discussions of measurement strategies that encourage increasingly sophisticated strategies and metacognitive thinking. Students need to actively participate in measuring activities and group discussions that involve using different units and pictorial representations.

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1. INTRODUCTION

Rational numbers are a complicated content area within school mathematics. Past research documented the difficulties of middle school students with various fraction and rational number concepts (e.g., Kieren, 1980; Behr, Lesh, Post, and Silver, 1983; Behr, Wachsmuth and Post, 1985). A possible reason for this difficulty might be that rational numbers can be interpreted in various ways, such as part-whole, ratio, operator, quotient, and measure (Kieren, 1976, 1980; Behr, Lesh, Post, and Silver, 1983; Behr, Harel, Post, and Lesh, 1992). One interpretation of rational numbers that students' have demonstrated difficulty with is measurement, specifically as it is applied to length measurement. The results of the National Assessment of Educational Progress (NAEP), for example, revealed that American middle school students have difficulties with certain measurement concepts, such as length and perimeter (Martin & Strutchens, 2000). According to the National Center for Education Statistics (NCES, 2003) only 39% of the eighth graders could estimate the length of one object using another correctly, only 21% could predict the perimeter of a quadrilateral using a given unit of length, and 33% could determine correctly how many boxes of tiles are needed to cover a given area. There are many possible explanations about this deficiency.

One that strikes us as especially plausible was suggested by Stephan & Clements (2003), that is, instruction of rational numbers as measures mainly focuses on procedures of measurement activity (e.g., how to measure) rather than the underlying concepts of measurement such as unit, unitization, unit iteration, partitioning, transitivity, and conservation. In this article we address the difficulties of participating middle school students' understanding of "unit" and "unitization" in the measurement interpretation of rational numbers, using the number line as a tool for inquiry and understanding. We showed what kinds of "unit" understanding are requisite for viewing rational numbers as measures and how classroom instruction can scaffold those prerequisite knowledges. In particular, we focus on building from students' predominant understanding of rational numbers as part-whole ratios and associated conceptions of units, unitization, and partitioning. We start with an explanation of part-whole, measurement and the measurement interpretation of rational numbers and then discuss the number line model. Next, we examine students' measurement thinking. Finally, we provide a classroom example of how seventh grade students developed an understanding of rational numbers as measures building on their part-whole interpretations of rational numbers using number lines.

1.1. Part-Whole, Measurement, and the Measurement Interpretation of Rational Numbers

Numerous researchers have identified different aspects of rational numbers, the two we mention here are part-whole and measurement (Kieren, 1976, 1993; Behr, Lesh, Post, and Silver, 1983; Behr, Harel, Post, and Lesh, 1992). Part-whole interpretation of rational numbers involves comparing parts to the whole in a "part-whole" ratio. For instance, in the expression *a/b*, *a* refers to *a* parts of a total *b* parts. Traditionally, teachers introduce part-whole representations of rational numbers focusing on two-dimensional geometric shapes partitioned into several equal parts (Mack, 1995).

Measurement, on the other hand, involves identifying attributes (e.g., length, area, weight, volume) of an object or phenomenon, selecting a unit, and comparing the unit with the attribute of the object or phenomenon (Stephan and Clements, 2003; van de Walle, 2006). For example, for a length measure, one needs to, first, specify the endpoints of an object and then a unit to quantify the distance between the endpoints of the object by iterating the unit alongside the object that is measured. Measurement as a general construct includes three main principles: (1) an inverse relationship between the size of the measuring unit and the number of times the unit is used to measure a given quantity; (2) the possibility of partitioning the unit into smaller and smaller units until one can approximate a given quantity with any desired precision; and (3) iterating the unit end to end alongside the object or phenomenon being measured (Stephan and Clements, 2003, pp.3-4). The measurement interpretation of rational numbers involves all aspects of measurement, such as specifying a unit, determining some length, and measuring the length with the unit via iteration (Lamon, 1999). For example the fraction 2/3 can be interpreted as a length of 2 iterations of a unit with length 1/3rd. The same principles of measurement apply to the measurement interpretation of rational numbers. The unit can always be subdivided into smaller units, generating a greater quantity of units that correspond to some desired specificity; and the iteration of unit is continuous process that starts from or is done in reference to some zero-point.

The concepts of partitioning, unit, and unitization are a common to both part-whole and measurement interpretations of rational numbers. Despite these commonalities, there are differences. In part-whole contexts, partitioning involves comparing the number of equal parts to the total number of equal parts, while in measurement "the number of equal parts in the unit can vary, and what you name your fractional amount depends on how many times you are willing to keep up the partitioning process" (Lamon, 1999, p. 113). Furthermore, in part-whole situations, students generally deal with one unit, such as partitioning a pizza into several parts (e.g., ½'s, 1/3rd's, or 1/4th's), or partitioning a box of soda among several people. In contrast, in measurement contexts students often deal with measures of multiple units. Due to similar language, sets of symbols, and similar representations it is likely that students will overgeneralize part-whole partitioning strategies in measurement contexts.

Among the important elements of rational numbers as measures is the concept of the unit. Units are used to determine numeric relationships between what is measured and the scale of measure (van de Walle, 2006). For example, to measure a length of an object, one needs to identify what the unit is and how many units are needed to match the length of the object being measured. However, discerning the unit is a challenging task for many students particularly when units are compared multiplicatively (van de Walle, 2006). Furthermore, unitization is the "cognitive assignment of a unit of measure to a given quantity" (Lamon, 1999, p.42). For example, when measuring the length of a pencil as 5 centimeters, the unit in this case is 1 centimeter, of which there are 5. If this same length is measured as 50 millimeters, the unit changes from 1 centimeter to 10 millimeter, and measurement changes by a factor of 10 (from 5 to 50), related inversely to the length of the unit. In this case it can be said that the length has been "reunitized". To distinguish differences between measurement and part-whole contexts clearly, and to show explicitly how these concepts are distinct within different situations, we used the number line model.

1.2. The Number Line Model

The number line is a practical model for introducing systems of units and ties directly to measurement. Specifically, the number line model has features different from other models (e.g., area and set models) in that it may be continuous while set models are visually discrete (Bright, Behr, Post, & Wachsmuth, 1988). Also, typically the distance from 0 to 1 on a number line represents the base unit and the model involves iteration of that unit and supports "simultaneous subdivisions of all iterated units" (Bright, Behr, Post, & Wachsmuth, 1988, p. 235). The analogical tie to distance, a physical attribute, is critical.

The rational number on a number line represents some distance from a zero point (Lamon, 2005). For example, 3/4th on the number line represents the distance from zero to 3/4 by three iterations of a 1/4th-unit. Also, other measurement principles can be represented with a number line. For instance, the distance from 0 to 3 can be represented as 3 (1-unit), 6 (1/2-unit), or 12 (1/4-unit). As the size of the unit decreases (e.g. from 1 to 1/2 to 1/4) the number of units needed to measure the same distance increases proportionally (e.g. from 3 to 6 to 12). Each smaller unit allows for a more precise measurement.

1.3. Students' Measurement Thinking

Students have traditionally demonstrated an aptitude in using measurement concepts. This aptitude, however, has been found in geometric contexts, rather than those with one-dimensional lengths, such as embodied by number lines (Payne, 1976). Studies examining students' measurement thinking using number lines are limited. Here we discuss these studies and their connections to this investigation.

Payne (1976) reported that elementary students demonstrated a greater ability to determine fractional parts of area models using part-whole strategies than locate fractions on number lines. Novillis

(1976) supported Payne's findings using middle school students. She reported that middle school students more easily located proper fractions in geometric regions than on number lines. She argued that students' difficulties were caused by the application of part-whole strategies to number lines from zero to one. She also found that students partitioned number lines in the same way as discrete areas, with the fraction representing some number of partitions, or parts, in the entire area/length, or whole. Fractions were, therefore, partwhole ratios rather than measures. Novillis further revealed that the length of the number line might be a critical factor in students' difficulties, saying that "whenever a number line of length one is used, then the number line model is not being completely tested. In this case, the number line is really just another part-whole model where the unit is not in question, being the 'whole'" (p. 423).

Bright, et al. (1988) discussed students' thinking about number lines using the number line from zero to one, and compiled a number of strategies and difficulties of students. In general, they found that elementary school students displayed difficulties with untization, reunitization, iteration, and associating fraction symbols with number line representations. In particular, "students were unable to choose a reduced fraction name when an unreduced equivalent form was represented on a number line" (Bright, et al., 1988, pg. 217).

The source of students' tendency to apply part-whole strategies to measurement tools such as number lines has been tenuously attributed to knowledge of whole numbers with discrete units (Behr, Wachsmuth, Post, & Lesh, 1984). Researchers found that judiciously designed tasks potentially aided the development of students' measurement concepts (Tzur, 2004).

While the number line can be used as a tool for measurement, part-whole interpretations of rational numbers can disrupt learning of measurement concepts. Given these complexities, and given that there has been little research studying the nature of students' understanding of rational numbers as measures in either individual or classroom teaching contexts, the present study attempts to broaden our understanding about students' changing ideas of unit and unitization associated with the measurement interpretations of rational numbers using number lines longer than one, and to identify instructional strategies that will help students develop such an understanding of rational numbers as measures. Of particular interest was how those instructional strategies might be used and useful in whole-class settings, and build on students' prior knowledge and tendencies to view fractions as part-whole ratios.

2. METHODOLOGY

This study was part of a larger NSF-funded study that investigated students' understanding of rational numbers that included individual interviews (student and teacher) and classroom observations.

2.1. Settings and Participants

Fifty-six seventh graders participated from three different classes in an urban K-8 school located in Arizona, U.S.A.. Participation in the study was voluntary and, according to the classroom teacher, the participating students represented various levels of performance in mathematics (e.g., low, middle and high). The school had predominately Hispanic students of lower-middle class background as over 90% of them received free or reduced lunch. The district adopted mathematics curriculum consisted of the National Science Foundation (NSF) sponsored, *Mathematics in Context* (MiC) (2003) and Arizona Instrument to Measurement Standards (AIMS) test preparation materials.

2.2. Procedure

A four-day classroom teaching experiment was conducted to investigate students' ideas of unit and unitization associated with the measurement interpretation of rational numbers in a whole-class instructional setting. The goal was to examine students' development and understanding of mathematical concepts (specifically measurement with number lines) by providing them mathematical tasks and content to which they have not been previously exposed. The overall cyclic process of data collection and analysis for this project was outlined as follows (adapted from Middleton et al., 2004):

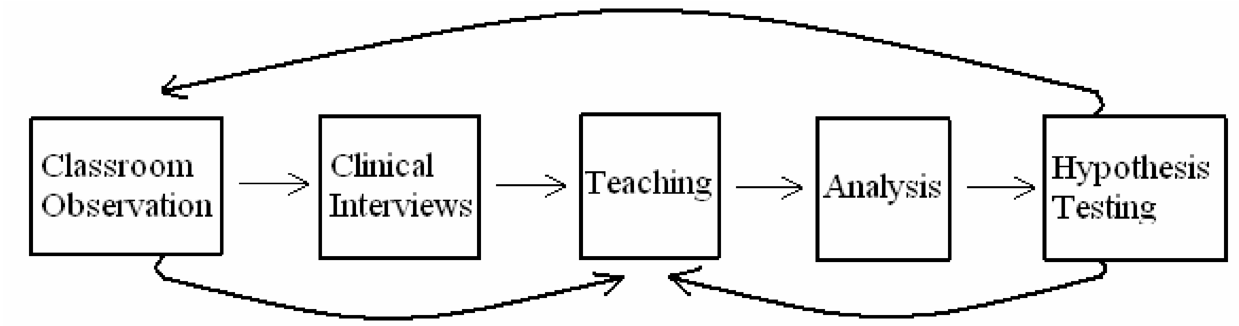

Figure 1. Overall cyclic process of data collection and analysis for the project (adapted from Middleton, et al., 2004)

Before instruction, based on a review of existing research literature on length measurement and number lines, we developed and administered multiple assessment tasks to the participating classes. The assessment asked students to locate both proper and improper fractions on number lines ranging from zero to one, zero to three, and zero to five. Some examples: students were asked to locate 1/2 on a number line from zero to five, and to locate 13/3 on a number line from zero to five (similar to the tasks used by Bright in 1988) to inform the instruction and aid students' learning of measurement and rational numbers.

Based on the assessment results, we identified students' initial difficulties, made hypotheses for possible sources of those difficulties and proposed an instructional intervention. We designed the instructional units based on these students' pretest results, and collaborated with the regular classroom teacher who taught the instructional unit with one of the researchers. Three 7th grade classes were taught consecutively but separately at the same day for four days. Each teaching session lasted in 80 minutes. During the teaching sessions, one of the researchers made observations and took field notes, hypotheses were tested and new hypotheses were generated, to revise and form subsequent days' instruction. After each classroom session, we worked with the classroom teacher, discussing and analyzing student thinking, generating and revising hypotheses, and designing the next day's instruction (see Figure 2 for the instructional design cycle).

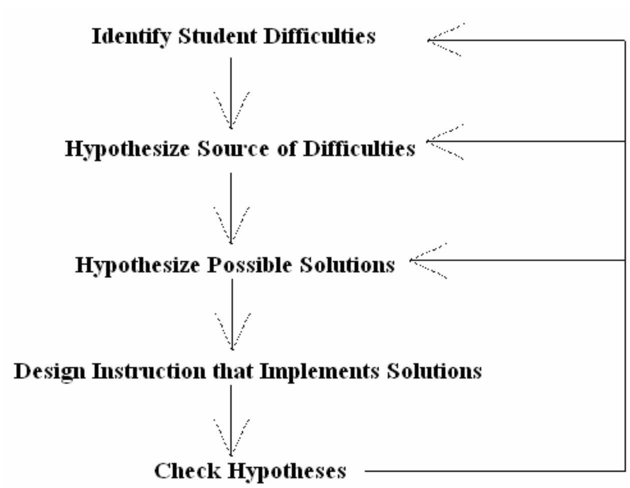

Figure 2 Instructional design cycle

Five students of varying abilities were chosen for individual, task-based interviews in order to investigate students' unit and unitization ideas on number lines. The students were chosen because their performance was typical of various degrees of sophistication throughout the three classrooms. The individual task-based interviews were performed in rooms separate from the main classroom; twice weekly, over three consecutive weeks, each interview lasted 45 to 50 minutes. The purpose of the task-based interviews was to more fully explore their understanding of rational numbers as measures on number lines, investigate the development of those ideas, and identify possible learning trajectories. Interview foci included students' written inscriptions and verbal "think aloud" responses. Interviews additionally focused on students' reasoning, procedures, and how their reasoning and procedures changed over time. Two of the five students eventually dropped out, and of the three remaining students Malcolm was chosen as a case because he demonstrated a clear trajectory in his development of ideas about unit and unitization on the number line. For the purpose of this study, we will focus on the whole-class teaching episodes.

2.3. Data Analysis

We analyzed students' responses by various sets of criteria. The first was the conventions of formal mathematics. How students located the given rational numbers on number lines allowed us to initially classify the full range of responses into "bins" that corresponded with various proportions of questions "right" or "wrong" (Miles & Huberman, 1994). When we decided that one question was typical of a set of questions, we used that question and the mathematical correctness of students' responses as a proxy for the overall percentage of correct responses to the similar questions. For example, many pretest questions asked students to locate proper fractions on number lines from zero to one. Instead of using the overall percentage of correct answers to those questions, we chose one question (locating $3/4^{th}$'s) as a placeholder for the entire subset of similar questions. Second, we analyzed students' responses within the aforementioned "bins" for idiosyncratic differences that would reveal subtleties in students' thought processes. Also, we isolated common misconceptions the students, on average, exhibited. They are described below.

When some student response was particularly confusing, we adopted the assumption that the student had made sense of the problem in some way. Our goal, then, was to suppose what type of understanding the student must have based his or her response on in order to answer the question in the way her or she did. Both commonalities (in superficial student response and in inferred student thinking) were used to design instruction and form conclusions. A similar method was used to analyze the posttest results and choose students for follow-up, task-based, individual interviews.

3. RESULTS

3.1. Experiment

While students' measurement difficulties were the initial impetus for performing the teaching experiment, we used an instructional design cycle to plan, implement, and evaluate instruction that emerged. The cycle of designing instruction went through several stages (see Figure 2).

3.2. Identify Students' Difficulties

Fifty-six seventh grade students from three math classes were pretested, based on existing literature concerning length measurement and number lines, to determine their initial understanding. Students demonstrated two primary misconceptions when using number lines.

3.3. Misinterpreting the Fraction Notation

First, fraction notation, with improper fractions, was read inverted to make proper fractions, resulting in a wrong answer. Jeffrey, for instance, misinterpreted the fraction 3/2 as its inverse, 2/3 (as shown in Figure 3). He divided the number line into three equal pieces, finding 3/2 at the location of 2/3 on the number line as if it were a number line from zero to one.

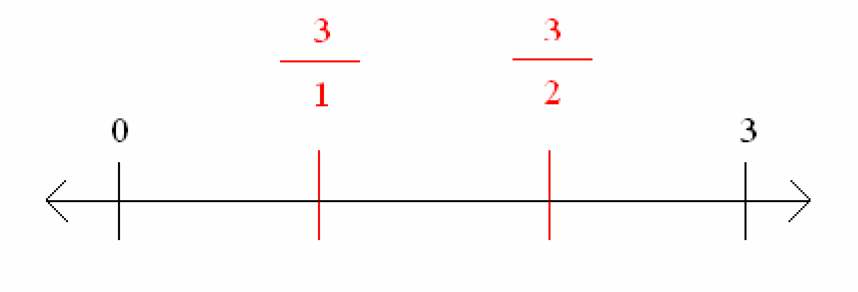

Fig. 3 Jeffrey's attempt to find 3/2 on a number line from zero to three

3.4. Misinterpreting the Whole Number Line as the Unit

Students also misinterpreted the number line as the unit rather than a composition of units. Using both number lines from zero to one and zero to five students considered the total given length of the number line as their whole or unit. If a number line was only from zero to one, students could visually partition the picture and obtain correct answers. When number lines represented a collection of wholes or units, part-whole unitization strategies were no longer appropriate. For example in Figure 4, Rene found ¾ by first dividing the whole number line into half and half again to find ¾. He used the number line from zero to five as a single unit rather than a collection of five, continuous units of length one.

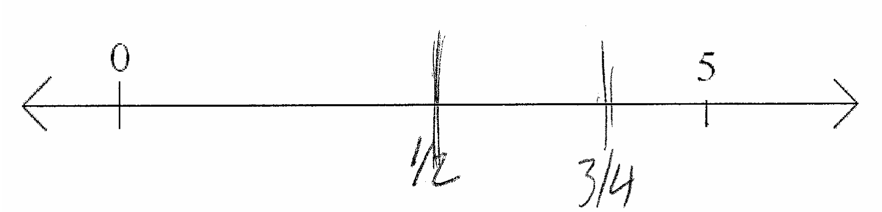

Fig. 4 Rene's attempt to find ¾ on a number line from zero to five

3.5. Hypothesize Source of Difficulties

In regard to the first misconception, we hypothesized that students' inversion of the fraction notation was an attempt to transform improper fractions, with which they were unfamiliar, into proper, more familiar fractions (see Figure 3).

In regard to the second misconceptions, we proposed that students interpreted ideas of unit and unitization in terms of part-whole ratios without accompanying measurement ideas or understanding of the number line. Students were unable to understand a number line as a continuous collection of units, or wholes. Instead they viewed the entire number line as the "whole" (1-unit), and misused the number line as a fraction bar.

3.6. Hypothesize Possible Solutions

Their incorrect interpretations of the fraction notation itself lead naturally to modeling the correct use and interpretation of fractional notation in general. To help students overcome misconceptions about measurement ideas of "unit" and "unitization", we hypothesized that instruction should focus on the application of part-whole ideas, properly extended into measurement contexts. We hypothesized that if students used the ideas of the unit and unitization in measurement contexts, they would begin to recognize the number line as a collection of continuous wholes.

3.7. Design Instruction that Implements Solutions

Instruction was designed based on student's initial understanding shown on pretest results and resulting hypothesized solutions to students' difficulties. While proper measurement interpretations of rational numbers requires that measures be continuous iterations of a unit, because the students displayed such difficulty in simply identifying the unit, the transitional activity was designed to bridge both measurement and part-whole interpretations. For example, because students viewed the entire given length of the number lines as the whole, the aim of the first day's instruction was to help students see a measure as a continuous length of wholes, rather than units per se. Instruction lasted four days in three different seventh-grade math classes. Each class contained a nearly equal portion of the fifty-six students.

3.8. Implementation of Instruction

On first and second day of instruction, students used Dr. Loyd's fraction kit that included wholes, halves, thirds, fifths, sixths, and eighths, using only the width of each piece. Students used the whole to measure the lengths of their tables in small groups and then drew their table measurements with models on transparences. Although the size of the tables were similar, students' measurements varied based on how carefully they measured (8 and ½ wholes versus 8 and 1/3 wholes). They began iterating the "wholes" to measure the lengths of their rectangular tables with various estimations of remaining partial iterations of wholes, or left-over lengths that did not fit into another whole. For instance one group found 7 ½ wholes as their measurement. Each small group presented in whole class discussion. Unlike part-whole situations, in which the whole constituted the unit, the whole applied in this measurement activity became the unit of iteration. The measurement of the table was therefore a length represented by some number of iterations of the unit, or whole.

Each group was asked to predict how their measures would change when using halves instead of wholes. Most students began estimating with their hands the number of halves that would measure the length of the table. Only one group was able to conclude that their measurement of the table using wholes could be multiplied by two to find the length in terms of halves without measuring. The following conversation between the researcher and a student reveals how this student thought about reunitization.

R: researcher; *S*: student

R: What did you measure?
S: 8 ½.
R: 8 ½ what?
S: 8 ½ wholes.
R: Okay. What would be the length of the table if you measured it with ½'s. Can you tell me without actually measuring with ½?
S: [paused, estimated two iterations with his hand] 16 ½.
B: 16 what?
S: 16 halves.
B: How did you get 16?
S: 8 times 2.
B: How many halves you have altogether?
S: 17.

The group in the above vignette was able to simultaneously subdivide each whole or unit into a smaller unit or whole to obtain an answer without manually iterating the new unit. The other groups displayed difficulty in determining the length of the table with halves accurately without manual measurement and ultimately began by estimating with varying accuracies. Students then used the half fraction strip to measure the length of the table, manually verify their predictions, drew pictures of their strategy on transparences and presented.

On the third and fourth day of instruction, because in previous days' lessons students expressed difficulty in estimating measures using units smaller than the whole, we hypothesized that students difficulties were sourced in not recognizing the inverse relationship between the length of the units and the number of units needed to measure some length. Accordingly, we used this as the goal of instruction. Students were asked to measure their tables using thirds, and predict the measurements using fourths, fifths, and eighths, and encouraged to verify their predictions with manual iteration. They were able to recognize that when the unit grows smaller the table measurement increases, without actually changing the total length of the table (8 1/3 with a unit of 1 whole versus 25 with a unit of 1/3). For example, Van found that the length of the table is 8 1/3 wholes, or 16 halves and 1/3 of a whole, then reunitized the entire amount (8 1/3 wholes) with a unit of 1/3 as follows:

R: It was 16 halves and then you had the third, so how many thirds would be in your table if you were to measure it in thirds?
S: 24 [thirds].
R: 24, how did you get that van?
S: Eight times three.
R: Ok so in eight there are 24 thirds so you got 8 and 1/3 so its…
S: 25 [thirds].

Next, we asked the entire class to measure number lines with the same pieces used to measure the tables. For example, a number line from zero to five was drawn and the class was asked what the measure of the number line would be using wholes, halves, thirds, and fourths. The largest difficulty in finding lengths along the number line was locating the first iteration of the unit. Students' confusion about the location of zero, or if iteration began at zero or one, prevented them from being able to perform that first iteration. Once students were able to see numbers on a number line as measures of distances, they began iterating from a zero point and expressed the length of several number lines with different units (e.g. 0 to 1, 0 to 3, 0 to 5 etc.). The goal of this activity was to complete the transition from using wholes as separate, discrete entities, to using wholes as units that can be iterated to measure distance. The number line was used to scaffold this transition.

Then, students were asked to show on a number line the measurement of the table in wholes, halves, and thirds. The purpose of this activity was to help students practice the iteration of units to find measurements using number lines, that is, to transition from the concrete representations of table measurements to the abstract representation of the number line. Group A drew a partial number line to show their method of measuring the table with different units, their table measurements using the wholes and halves that they later represented on the number line are shown in Figure 5.

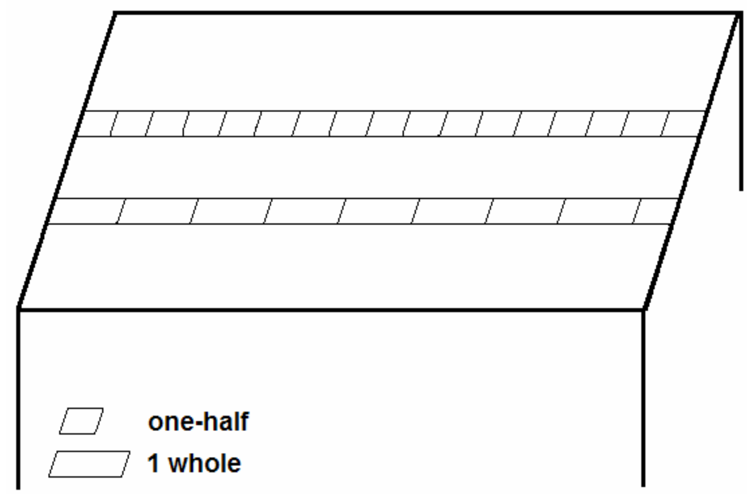

Fig. 5 Group A's method of measuring the table using wholes and halves

Students, on the last day of instruction, were able to see a number line as a collection of units of length one. They, when given an empty number line from zero to five, suggested that a "length of one" should first be labeled. They added another length of one to the distance from zero to one, and so on until the entire number line was unitized. This showed that students saw the units on the number line as continuous and connected lengths of one, and the need for iterating units on the number line when attempting to measure some length. Moreover, students began to conceptualize the number line as different from fraction bars and as a tool for interpreting rational numbers as measures.

However, students had difficulties identifying the unit of improper fractions. For instance, when asked to locate 9/4 Jeff initially was unable to recognized the unit as ¼, instead representing 9/4 as a mixed number by assuming a unit of one, and finding how many wholes could fit into 9/4 with some remainder. He then reunitized his wholes in terms of 1/4 and added to obtain the original 9/4 (see Figure 6).

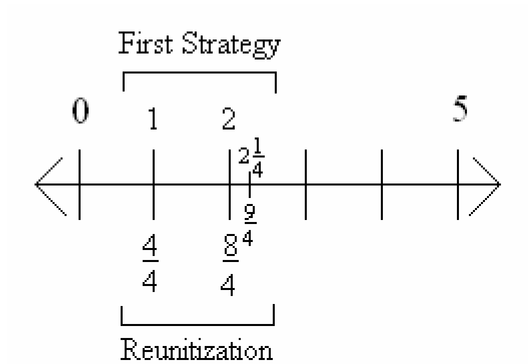

Fig. 6 Students' strategies for locating 9/4ths on a number line from zero to five

3.9. Emerging Ideas and Strategies

After instruction, several strategies for correctly interpreting fractions as measures emerged: repeated halving entire number lines, repeated halving pre-unitized number lines, and iteration of given units. Also, no student misinterpreted (by inverting improper fractions) the fraction notation on post-test results.

3.10. Repeated Halving of Entire Number Lines

Repeated halving was a strategy that students were able to use to correctly locate fractions on number lines (see Figure 7). Jessica, when asked to locate ¾ on a number line from zero to five, cuts the entire number line into two pieces, then the first half again into half to find 1 ¼. She used this length to locate one. She then divided her one-unit into four pieces to find her unit of ¼ and iterated it three times to find ¾.

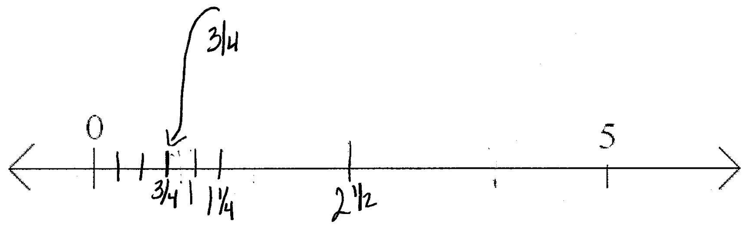

Fig. 7 Jessica's repeated halving strategy

3.11. Repeated Halving Pre-unitized Number Lines

Repeated halving was also used after students first divided the number line into units of one (see figure 8). Carlos pre-unitized the number line into five one-units then subdivided zero to one in half to find 1/2, then again to correctly locate a measure of ¾.

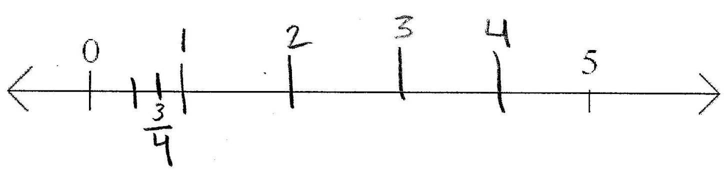

Fig. 8 Carlos's Halving Strategy

3.12. Iteration of Given Units

Students also iterated the given unit to find lengths measurement. Students were able to recognize the unit of ½ in the fraction 5/2, and iterate that unit of ½ five times to correctly locate 5/2 on the number line (see Figure 9).

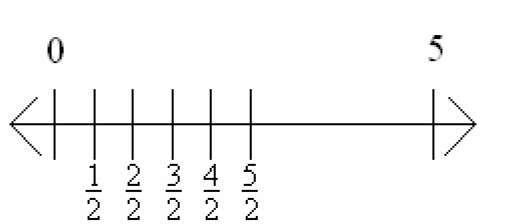

Fig. 9 Students' iteration strategies of ½ to find 5/2 between zero and five

3.13. Pre-Posttest Gains

While students performed better on every assessment given to them, these gains in student achievement were potentially spurious for a number of reasons. First, students may have learned how to take the test rather than the content per se. Second, no psychometric evaluation of the test itself was performed. Third, the assessment, both pre- and posttest were intended only as previously described. At no point were they intended to measure student accuracy in an answer-oriented sense. Our focus was on the strategies around answers, and insofar as students answered similarly, we reasoned, their strategies were potentially similar as well. Fourth, our goal was to qualify student thinking, rather than quantify student difference scores. As researchers have pointed out, pre-posttest designs do little to qualify student learning. Because including pre-posttest scores does little to qualify students' thinking on the measurement tasks given to them, and because casual links between instruction and student gains could not be differentiated from other plausible sources of change in students' performance, we omitted these results.

4. DISCUSSION AND CONCLUSION

Students' difficulties in applying rational numbers as measures on number lines revealed that the unit and unitization concepts do not develop naturally. Moreover, when using an abstract tool such a number line students did not easily see fractions as measures of distance along those number lines. Using concrete materials as a unit and focusing on the iteration of the unit helped students develop a measurement "sense". Transitioning from concrete representations to more abstract representations (e.g. number line) is critical to fully understanding rational numbers as measures.

This point should be repeated. Students' understanding of "unit" as a concrete, length measurement is critical to their understanding of rational numbers as measures. When units remain nested within well developed part-whole instructional histories, students exhibit a number of misconceptions concerning the use of rational numbers as measures (see above results for specific misconceptions students had). When length measurement was concretized with a physical object of some kind, in our case a fraction strip, and that object was iterated continuously to measure some distance, in our case their tables, students were able to connect the broad concept "unit" within measurement contexts.

Reunitization, or in our case students' projecting length measures with various units independent of physical iteration, allowed those students two realizations. The first was that the size of the unit was inversely related to the number of units involved in some measurement. Second, and more important to rational numbers as measures, it that the unit became a fraction of one, as denoted by the denominator. The number of iterations, then, composed the numerator. The combination of this new understanding of numerator and denominator, and of the inverse relationship between size of unit and quantity of measurement (yielding increasingly accurate measurements) constituted, by definition, an understanding of rational numbers as measures. Students' representations provided a direct link between the quantities of their measurements, fraction notation, and unit size. Posttest results support the conclusion that students had gained an understanding of rational numbers as measures.

Several instructional strategies helped students see rational numbers as measures. Effective instruction should focus on the important concepts within measurement such as unit, unitization, and the iteration of units; and connect key mathematical content to students' previous knowledge and experience. Also, group activities need to actively engage children in meaningful representations and discussions of measurement strategies that encourage increasingly sophisticated strategies and metacognitive thinking. Students need to actively participate in measurement activities and group discussions that involve using different units and representing their models of measurement with pictures.

Measurement is an important aspect of interpreting rational numbers. Students' predominant reliance on rational numbers as part-whole relationships need not confound their understanding of rational numbers as measures as well. In fact, part-whole knowledge of rational numbers, as we have shown, can aid in students' development of rational numbers as measures. When coherent frameworks of instructional feedback, evaluation, and revision are combined with adaptive instruction, sensible, cognitive evaluation, and critical analysis, students' natural knowledge of part-whole ratios can support their developing awareness of the more complex relationships and potentials of rational numbers.

REFERENCES

Behr, M., Lesh, R., Post, T., & Silver E. (1983). Rational number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-125). New York: Academic Press.

Behr, M., Wachsmuth, I., Post T., & Lesh R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Behr, M., Wachsmuth, I., & Post, T. (1985). Construct a sum: A measure of children's understanding of fraction size. *Journal for Research in Mathematics Education*, *16*(2), 120-131.

Behr, M., Harel, G., Post, T., Lesh, R. (1992). Rational number, ratio and proportion. In D.Grouws (Eds.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: McMillan Publishing.

Bright, G., Behr, M., Post, T., & Wachsmuth, I. (1988, May).Identifying fractions on number lines. *Journal for Research in Mathematics Education., 19*(3), 215-232

Hiebert J., & Tonnessen, L., (1978). Development of the fraction concept in two physical contexts: An exploratory investigation. *Journal for Research Mathematics Education 9*(5), 374-378.

Kieren, T.E. (1980). The rational number construct: Its elements and mechanisms. In T.E. Kieren (Eds.), *Recent research on number learning* (pp. 125- 150). ERIC/SMEAC, Columbus, OH.

Kieren, T.E. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Eds.), *Number and measurement* (pp. 101-150). ERIC/SMEAC, Columbus, OH.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: an integration of research* (pp. 49-84). Hillsdale, NJ: Lawrence Erlbaum.

Lamon, S. (1999). *Teaching fractions and ratios for understanding* (1st Ed.). Mahwah, NJ: Lawrence Erlbaum Associates.

Lamon, S. (2005). *Teaching fractions and ratios for understanding* (2nd Ed.). Mahwah, NJ: Lawrence Erlbaum Associates.

Mack, N. (1995). Confounding whole-number and fraction concept when building on informal knowledge. *Journal for Research in Mathematics Education, 26*(5), 422-441.

Martin, W., & Strutchens, M. E. (2000). Geometry and measurement. In E. A. Silver (Eds.), *Results of the 1996 NAEP mathematics assessment*, (pp. 193-234). Reston, VA: NCTM, 2000.

Middleton, J.A., Flores, A., Carlson, M., Baek, J. M., & Atkinson, R. (2004). *A longitudinal study of the development of rational number knowledge in the middle grades.* Proposal submitted to the National Science Foundation. Arizona State University, Tempe, Arizona.

Miles M., & Huberman, M. (1994). *Qualitative data analysis: An expanded sourcebook*. Thousand Oaks, CA: Sage.

National Center for Education Statistics (NCES) (2005). "The nation's report card: Mathematics 2003 major results." [Online] Retrieved on September 14, 2005, at URL: http://nces.ed.gov/nationsreportcard/mathematics/results2003/.

Novillis, C. (1976). An analysis of the fraction concept into a hierarchy of selected subconcepts and the testing of the hierarchical dependencies, *Journal for Research in Mathematics Education, 7*, 131-144.

Payne, J. N. (1976). Review of research on fractions. In R. Lesh (Eds.), *Number and measurement* (pp.145-188). Athens, GA: University of Georgia.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children, *Journal for Research in Mathematics Education 14*, 307-317.

Stephen M., & Clements. D.H. (2003). Linear and area measurement in prekindergarten to grade 2. In D.H. Clements & G. Bright (Eds.), *Learning and Teaching Measurement 2003 Yearbook* (pp. 3-16). Reston VA: National Council of Teachers of Mathematics.

Tzur, R. (2004). An integrated research on children's construction of meaningful, symbolic, partitioning-related conceptions and the teachers' role in fostering that learning. *Journal of Mathematical Behavior, 18(2),* 123-147.

van de Walle, J. (2006). *Elementary and middle school mathematics: Teaching developmentally* (6th Ed.). Boston, MA: Allyn & Bacon.