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Mathematical knowledge of two middle school mathematics teachers in planning and teaching pattern generalization

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Abstract: Examining the knowledge of teachers in practice may shed light on understanding how students learn and finding out why they have difficulty in learning. This paper focused on teachers' knowledge of pattern generalization in instruction with the consideration of students' generalization strategies in planning. The multiple-case study design was used for this study to compare and contrast two middle school mathematics teachers' lesson planning and instruction. Lesson plans, pre-observation interviews, observations and post-observation interviews were used as the data collection tools. The data were analyzed by using the Mathematical Knowledge for Teaching (MKT) model. The findings showed that the two teachers used numerical reasoning in all representations, but they had difficulties in using figural reasoning. They generally used the tabular representation effectively to underlie the relationship of generalization. While one of the teachers defined the pattern concept correctly and always emphasized analyzing the relationship between the position number and the term, the other teacher defined the pattern concept partially correctly, and her inadequate explanations of functional thinking caused some misunderstandings of students about generalization. It was also observed that the students' lack of knowledge about algebraic expressions prevented them from obtaining a general rule. Through the cases of these two teachers, it was noted that teachers need to have a good conceptual mathematical understanding and also knowledge of students' thinking to design effective lessons.

Keywords: Lesson planning, mathematical knowledge for teaching, pattern generalization, students' generalization strategies

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INTRODUCTION

Practices are defined as "core activities that could and should occur regularly in the teaching of mathematics" (Franke, Kazemi, & Battey, 2007, p. 249). The practice routines of mathematics teaching need explaining in terms of content and grade levels. Besides, how the practices assist teachers' teaching of mathematics and affect students' learning should be analyzed (Ball & Forzani, 2009). Several studies showed appropriate practices, which provide doing mathematics within the context of problem-solving, constructions, and discourse, and supported students' learning (Arcavi, Kessel, Meira, & Smith, 1998; Chapman, 2006; Schoenfeld, Minstrell, & van Zee, 1999; Silver & Smith; 1996). Thus, Saxe, Gearhart, and Seltzer (1999) suggested developing teachers' knowledge to reveal effective practices because one of the factors that influence on teaching practices is teacher knowledge (Hiebert, 1997). In this regard, the focus of the current study is the knowledge in practice to examine how teachers use their knowledge in their teaching.

As Ball (2000) pointed out the gap between teacher knowledge and practice in mathematics generally, Doerr (2004) supported this claim for algebra teaching practices. Doerr (2004) defined this situation as a dilemma that referred to the contradiction of the knowledge with the practice in teaching algebra. Since teachers did not have conceptual perceptions and experiences from their learning, it caused their practice not to be effective. Thus, the researchers give suggestions to support and develop teachers' practices of algebra. To illustrate, Brown and Smith (1997) suggested the use of the questioning technique to help the students to think algebraically. Particularly, the presenting of tasks with algebraic thinking features, supporting teachers with experiences that included using models or representations of algebraic expressions, and showing how arithmetic and algebra can be connected can help improve teachers' knowledge and practices of algebra (Asquith, Stephens, Knuth, & Alibali, 2007; Ayalon & Even, 2015; Blanton & Kaput, 2001). Although there have been studies about algebra knowledge, Wilkie (2014) has asserted that there are fewer studies about teachers' algebra knowledge as topic-specific in practice in middle school grades. Therefore, the main issue addressed in this paper is teacher knowledge in teaching pattern generalization. The following research question is addressed, what is the nature of middle school mathematics teachers' mathematical knowledge for teaching pattern generalization in instruction with the consideration of students' generalization strategies in planning?

Related Literature

Teacher knowledge in practice

Adoniou (2015) explains 'knowing how' as using knowledge in practice as one of the components of teacher knowledge. However, Ball (2000) asserted that there was a gap between content knowledge and the practice of a teacher. Teachers' content knowledge is important to evaluate the students' ideas, and it helps present different and valuable opportunities about mathematics for students, but the presentation of content knowledge in practice can enable students to learn. Besides having strong and conceptual content knowledge, teachers should also know the relationship between the contents to understand the students' thinking and combine content knowledge with pedagogy as pedagogical content knowledge (PCK) to convey their knowledge to the students. In this regard, what content teachers should know for teaching, and how teachers learn to use this knowledge in practice are essential components for teaching (Ball, 2000). Franke et al. (2007) stated the importance of the knowledge of students' thinking to develop teachers' practices. Hence, the knowledge of students' understanding, as well as conceptual knowledge, is necessary to organize effective lessons pedagogically (Fennema & Franke, 1992; Hiebert, 1997; Hollins, 2011).

Ball and Bass (2002) described the term mathematical knowledge for teaching based on Shulman's (1986) concept of PCK. Ball, Thames, and Phelps (2008) developed mathematical knowledge for teaching (MKT) model that was more extensive than previous models. To describe the model, Ball et al. (2008) analyzed the teachers' practices and identified the mathematical demands of teaching as the descriptors of the model components. Thus, it is specific to mathematics teachers and a practice-based approach to the knowledge concept. The model is a domain map that shows mathematical knowledge for teaching and consists of subject matter knowledge (SMK) and pedagogical content knowledge (PCK) as in the below figure (Ball et al., 2008). Subject matter knowledge is subdivided into common content knowledge (CCK) and specialized content knowledge (SCK). CCK is "mathematical knowledge that is used in teaching, but not directly taught to students" and used by those who work with mathematics. SCK, on the other hand, is specific to mathematics teachers (Hill, Sleep, Lewis, & Ball, 2007, p. 132). According to Ball et al. (2008), PCK consists of knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). Teachers must know the students' common conceptions and misconceptions, errors, difficulties specific to a mathematics topic (KCS); know how to design instruction for teaching mathematics, make decisions about the instruction, lead a discussion, or evaluate a teaching method or technique (KCT); and have the knowledge of the contents regarding the curriculum order, suggested activities, important explanations for teaching, and the relation between the topics and grades (KCC). Ng (2011) noted that the MKT framework enables examination of teacher knowledge in the practical aspect. In this regard, this was one of the reasons for selecting the MKT framework, since this study attempted to reveal teachers' MKT for pattern generalization during teaching. Another reason was related to the MKT model's sub-domains and their detailed descriptions. Teacher knowledge is explained into sub-domains specific to mathematic teaching with the MKT model (Hill, Ball, & Schilling, 2008). Thus, it is considered to reveal existing teacher knowledge as specific to pattern generalization, which gives detailed descriptions of the knowledge domains. Development of teacher knowledge forms with including the knowledge of students' thinking for teaching topics, which is important work (Kanes & Nisbet, 1996). This study aimed to do this work, and what is more, it could serve topic-specific knowledge forms for algebra by focusing on pattern generalization. Another reason for selecting this model was that the researchers used mainly observations of teachers' instructions while developing it. Since the main data of this study included observations from instruction processes, the MKT model was found to be appropriate to investigate both content knowledge and pedagogical content knowledge.

Students' generalization strategies of patterns

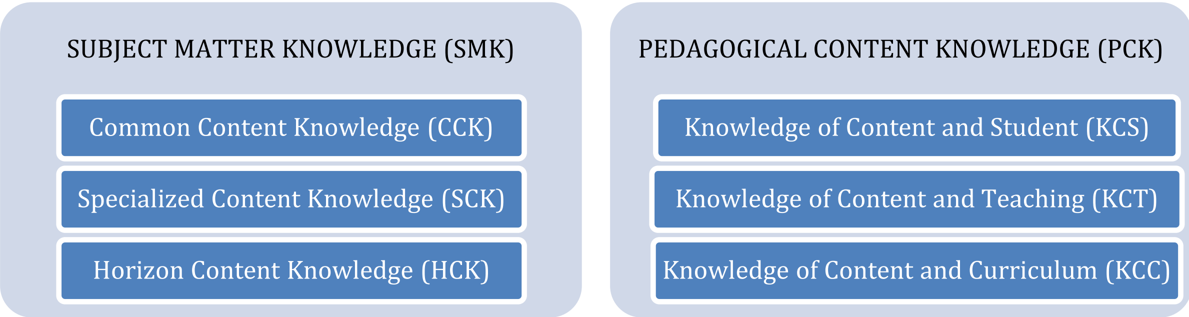

FIGURE 1. *Mathematical knowledge for teaching model (Ball et al., 2008)*

It is important to support the students to think algebraically in elementary grades in order to prevent middle and high school students' difficulties in algebra (Cai, Ng, & Moyer, 2011). Students begin to use algebraic symbols and notations in learning of generalizing patterns in school algebra. Therefore, pattern generalization is important since it is the beginning of formal algebra and so it can provide an understanding of the variable concept. Generalizing patterns also gives the idea of functional thinking with input-output relationships for early grade students (Hoyles, Threlfall, Frobisher, & Shorrocks-Taylor, 1999). Healy and Hoyles (1999) define recursive and explicit strategies for generalization. Students using the recursive rule that explain the relationship focusing on the difference among consecutive output values in the pattern. The explicit rule is finding an algebraic rule relating to input and output values in the pattern. Even students use both recursive and explicit strategies, students are expected to think explicitly to conceptualize generalization. Rivera (2010) explained this generalization process. Students seek the relationship in the pattern and try to propose a hypothesis based on the given steps of the pattern in abductive-inductive action. Then, they can extend the pattern regarding the hypothesis. To illustrate, exploring a relationship based on the total number of toothpicks in the 1st, 2nd, and 3rd step of the pattern and finding the number of toothpicks in the 10th step. This exploration is transformed into a rule as an algebraic representation in symbolic action (Rivera, 2010).

In the generalization process, the use of diagrams, tables, spreadsheets, and figures (figural patterns) to help students for interpreting the relationships between the input-output values of patterns (Lannin, Barker, & Towsend, 2006; Steele & Johanning, 2004; Warren & Cooper, 2008). Besides using these representations, students use figural and numerical reasoning to be able to generalize patterns algebraically (Walkowiak, 2014). However, many studies about pattern generalization have shown that students generally have difficulty in generalizing patterns algebraically (e.g. Becker & Rivera, 2005; El Mouhayar & Jurdak, 2016; Warren & Cooper, 2008). Students could generalize near generalization type of tasks in linear patterns; however, they had difficulty with getting a general rule for the n^{th} term in far generalization type of tasks. In this study, it was considered that the reasons that caused the difficulties of students could be explained by examining the teachers' knowledge and teaching pattern generalization.

Teaching pattern generalization

Teachers have an important role in teaching algebra since they construct the students' knowledge of algebra in early grades (Malara & Navarra, 2009). More particularly, the generalization of patterns for the transition of arithmetic to algebra is important in algebra (English & Warren, 1998). The studies related to generalization patterns showed that teachers had different conceptions about pattern generalization in algebra teaching. Bishop and Stump (2000) concluded that elementary and middle school teachers' conceptualizations for pattern generalization tasks were as problem-solving merely. They did not consider pattern generalization as facilitating of transition to algebraic thinking. When they were asked to identify students' strategies, they did not have adequate knowledge to explain them because their strategies were limited. The teachers' expectations did not match with students' strategies for generalization of patterns. To illustrate, the teachers expected the students to use functional thinking with co-variational strategies easily, but the students listed the terms to find near-term and they had difficulty to use functional thinking when the generalization was asked (Baş, Çetinkaya, & Erbaş, 2011; Blanton & Kaput, 2001; El Mouhayar & Jurdak, 2013). According to Wilkie (2014), although teachers have knowledge of content for pattern generalization, they do not have adequate pedagogical content knowledge for teaching this algebraic topic. More particularly, teachers could generalize patterns with words or calculation, and exemplify correct students' answers; but they could not explain strategies as recursive or explicit of students' answers. They also had difficulty in using algebraic symbols for general rule since they had a lack of relational understanding of functional thinking. The teachers did not have adequate experiences to create activities for teaching pattern generalization. Therefore, they could not use effectively function machine, or tables, or input-output values for teaching functional thinking.

These studies investigated teacher knowledge once a time, such as teachers' investigation on actual students' solutions, expected students' answers, or an instrument with open-ended questions. These findings could serve static data about teacher knowledge. However, few studies examine teacher knowledge of generalization of patterns in practice that is in teaching (Baş et al., 2011; Doerr, 2004; El Mouhayar & Jurdak, 2013; Wilkie, 2014). Accordingly, this paper attempts to explore teacher knowledge on the generalization of patterns in practice within a whole picture for teaching that is a process that includes lesson planning and reflections after teaching.

METHODOLOGY

Research Design

The current study focused on two teachers' knowledge in a process in the context of qualitative research. Kahan, Cooper, and Bethea (2003) suggest conducting qualitative research with different teachers to determine teacher knowledge as it affects the teaching process. With the current study, the teachers' knowledge in a process could give us more detailed information about their knowledge and also reliable patterns as the study was focused on a long process. Case study is especially preferred for this study in the light of the explanations regarding the characteristics of the qualitative research. Case study approach includes a case/cases that is/are explored by the researcher, detailed and in-depth data collection from multiple sources (e.g. observations, interviews), and reporting the themes for the case (Creswell, 2007). Particularly, multiple-case study design with a single unit of analysis was used from the types of case study designs (Yin, 2003). In the design of this study, the context was pattern generalization instruction, the cases were two middle school mathematics teachers, and the unit of analysis was the teachers' mathematical knowledge for teaching.

Participants

The participants were two middle school mathematics teachers who were working in the same public school. The names of the participants were changed for confidentiality, and code names were used as İpek and Gül to represent them. These teachers volunteered to participate in this research. Thus, these teachers were selected and convenient sampling method was used because of this accessibility. On the other hand, this selection could be a purposive sampling as it gives rich and detailed information about the teaching process of the teachers. The teachers in this study had a bachelor's degree in elementary mathematics education and graduated from faculty of education, and a master's degree in elementary mathematics education and they have also been in a doctorate program in elementary mathematics education. These teachers had more knowledgeable about research studies than other teachers with only a bachelor's degree. İpek has experienced in teaching middle school mathematics and has been working in a middle socioeconomic level school for 3 years. She has been teaching 5th, 6th, and 7th-grade students and thus she has been teaching algebraic topics for each level of the middle school for 3 years. Gül has been teaching 5th, 6th, and 7th-grade students for 9 years and thus she has been teaching algebraic topics for each level of the middle school for 9 years.

Data Collection

After all ethical permissions were obtained, the data were collected. The data collection procedure had two main phases in this study: pre-instruction and during-instruction. In the pre-instruction phase, the teachers prepared the lesson plans individually, and the researcher interviewed each teacher. In the second phase, during-instruction, the researcher observed the lessons in two different 7th grade classes, took field notes, and recorded video with a camera (Figure 2). After each class session, the researcher conducted post-observation interviews with the teachers.

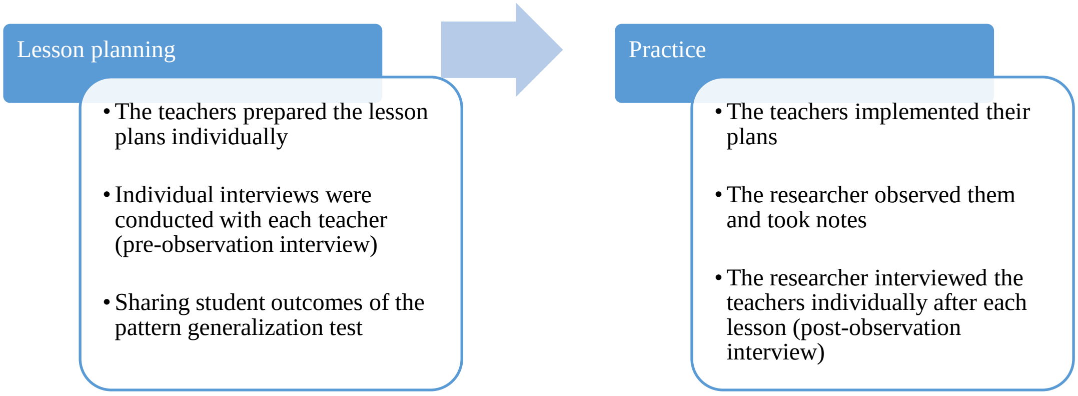

FIGURE 2. *The process of data collection*

The data collection process took about 9 class-hours (1 class-hour was 40 minutes). In this process, the observation of the instructions and the interviews with the teacher could give an accurate picture of teacher knowledge. While the planning provided an understanding of the teachers' existing knowledge, the instruction provided an observation on how the teachers used their knowledge in teaching. The purpose of the post-observation interviews was to check whether the researcher understood the teacher's instruction properly by observing the lessons by comparing the teachers' responses. Thus, the data collection process of this study could give a holistic understanding of the teaching with planning and instructions.

The context of the study

At the time of the data collection of this study, the teachers used the same textbook by the Ministry of National Education (MoNE) (2014) including the teacher's guidebook, students' textbook and students' workbook. They designed their lessons based on the textbook. The textbook had examples of linear and non-linear growth patterns, and the teachers used them. In the teaching process, the teachers generally asked questions, got answers from the students and evaluated the responses by discussing them with the class. The students generally worked individually in the class, and thus, they sat to see the board. They were 7th graders and 12-13 years old. The students had prior knowledge related to algebra. They had learned the concept of pattern and linear growth pattern generalization, and the concept of algebraic expressions in the 6th grade. The research was conducted in two classrooms, and there were about 33-35 students in each classroom. Each classroom had a projector and a whiteboard. However, the teachers did not use the projector throughout the observations. Although there were bulletin boards in the classrooms, there were no mathematics works of the students.

The setting of lesson planning

The lesson plans were prepared by each teacher individually based on the objective 'represent the relation in number patterns which are modeled with figures algebraically using variables' in the curriculum before the instruction. Then, the researcher and the two teachers came together and the researcher presented the answers of students in a test related to pattern generalization. This pattern generalization test was developed by the researchers in the previous semester using the questions in the literature and they implemented the test to 6th, 7th and 8th-grade students in the same school before this study (Girit & Akyüz, 2016). The test contained numeric, figural, and tabular representations of linear growth patterns as six tasks that were adapted from the literature (Blanton & Kaput, 2003; Lannin et al., 2006; Magiera, van den Kieboom & Moyer, 2013; Moss, Beatty, Barkin, & Shillolo, 2008; Stacey & MacGregor, 2001; Warren & Cooper, 2008). One of the tasks from the questions was as in the figure 3. The purpose of sharing the findings of the test was to make the teachers aware of the students' generalization strategies by examining actual students' solutions. The findings showed that 7th graders could write algebraic expressions but most of the written expressions were not correct since they only focused on the difference between output values. The students who could not generalize algebraically expressed the relationship verbally. They also used numerical reasoning in all types of patterns in the test, numerical, figural, and tabular representations of linear growth patterns. They usually seek a relationship between numbers. They counted the units in figures and they used numbers to get the general rule of the figural patterns. These students' algebraic generalizations could be supported by guiding them to use figural reasoning in figural pattern contexts (Girit & Akyüz, 2016). Another important finding was that the students who got the correct generalization generally used tabular representations. They transformed the values in the patterns to a table and they could investigate the functional relationship between input and output values easier in the table (Girit & Akyüz, 2016). While the teachers were examining the students' reasoning and generalization strategies in the answers, they decided to use tables for showing the functional relationship, and figural patterns for supporting the figural reasoning of the students who had difficulty in generalization by using numerical reasoning. The teachers considered these implications while revising their plans. Therefore, this study aimed to examine the teachers' knowledge in practice with their consideration of students' generalization strategies.

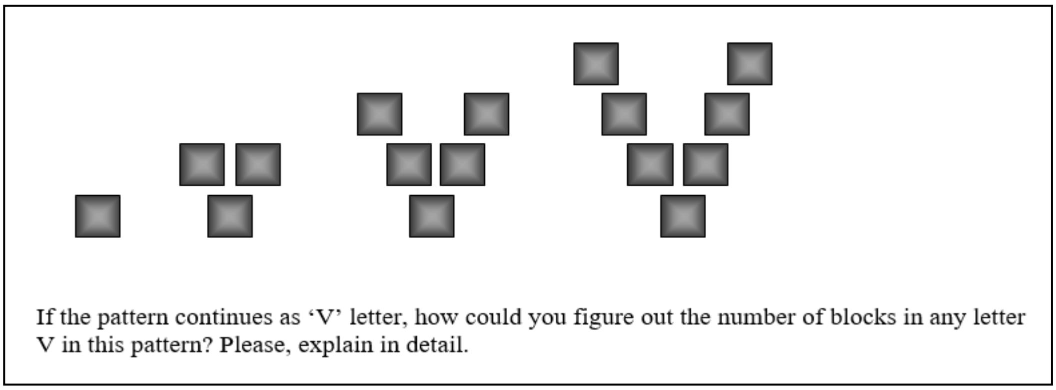

FIGURE 3. *One of the figural linear growth patterns in the test (adapted from Magiera et al., 2013).*

Data Analysis

The analysis of the qualitative research designs begins with forming the initial and tentative coding and then continues with grouping them in themes concerning similarities and ends with reporting the data (Merriam, 2009). For this study, the cases as the two teachers' planning and instructions were analyzed and described independently first. The data from the cases were analyzed and coded as a statement, an explanation, a dialogue, or a question that was considered meaningful. Then, the extracted codes were put into one of the themes and then sub-themes considering the descriptions and definitions of themes and sub-themes in the MKT model. SMK and PCK as knowledge domains were considered as themes and their components as knowledge sub-domains (CCK, SCK, KCS, KCT, and KCC) were considered as sub-themes for this study since the purpose of the study is to examine teachers' MKT. This process provided an analysis of cases individually within itself. Merriam (2009) stated that "a within-case analysis is followed by a cross-case analysis" (p. 205) for the analysis of multiple case studies. Thus, after forming these tentative codes, the codes which were categorized by comparing and contrasting and had a pattern in one case were also analyzed for the other case. To ensure the reliability of the research, the data were coded separately by the researchers. Then the researchers came together to evaluate their coding and then reached a full agreement with discussing the coding. Besides, in this study, data were collected from several sources that were preparing lesson plans, interviews, and observations to support trustworthiness.

FINDINGS

In this section, the MKT of İpek and Gül are presented within two sections: *planning* and *instruction and reflections* concerning the phases of data collection. The *planning* section includes the findings related to the content of the teachers' lesson plans and their explanations from the pre-observation interviews. The *instruction and reflections* section includes the findings related to the teachers' instruction of pattern generalization with their questions, responses to the students, explanations about the pattern generalization process and reflections from the post-interviews.

The Case of İpek

Planning

To start the instruction, İpek planned to begin with simple linear growth figural patterns that could be represented with numbers like 5,10,15… or 4,8,12… The examples were linear growth figural patterns with their general rules as 3n, 5n and 4n, and these could be written using variables only and not require any constants. Then, she planned to ask linear growth patterns whose general rules required a constant such as the patterns 3,5,7,9…, or 4,6,8,10... The last activity was a non-linear growth figural pattern (2,6,12,20…) with formed unit cubes. In this regard, the teacher sequenced the examples and activities to teach pattern generalization from simple to complex ones, and these were appropriate for the level of the students. This showed her appropriate teaching knowledge to design the instruction of pattern generalization (KCT). The teacher chose figural patterns and represented the terms with manipulative in particular. Her explanation related to using figural patterns showed her appropriate knowledge of the students. She expected that using figural patterns would be interesting and motivating for the students to understand the relationship in the pattern. Her knowledge of the students appeared appropriate (KCS). She also indicated that the figures could help the students to explore the increment or growth between the terms. She planned to ask the students to work in pairs or small groups to represent the number of units in figures in these patterns as numerical within a table. She explained that, in a table, the students could see the arithmetical relationship such as 3x1, 3x2, 3x3 ... one under the other in rows, and it could help the students write the general rule algebraically easier by recognizing what changes were, and where *n* should be written. Her specialized content knowledge to link representations (figural, numerical and tabular) to the underlying functional relationship appeared appropriate (SCK). She also emphasized that graphical representation could be useful for the students to show the change of the terms, but she could not explain the use of this representation for the pattern.

Instruction and reflections

The findings of the instruction process are presented under three headings as *beginning*, *class discussions* and *explanations* to understand the teacher's knowledge of pattern generalization. The reflections are given where relevant to the situation.

Beginning

First, the teacher asked questions about the pattern concept, such as what it was, and how it was formed. İpek's developed definition for the pattern concept with using students' answers appeared as "… the patterns are the relations between the different figures, aren't they? We will do operations with numbers in patterns." In this situation, her knowledge to develop a usable definition appeared inadequate because she defined the pattern as the relations between the figures, but the patterns were also created with numbers and had relations in them. Thus, her developed definition could be inadequate and might cause a lack of understanding as if patterns always had figures. At that point, she tried to connect this definition with numbers, but she stated the operations with numbers this time. She would also state using variables and algebra to increase the students' awareness of pattern generalization. Her specialized content knowledge to develop a usable definition of pattern appeared inadequately (SCK).

After reminding the pattern concept and giving the definition of it, İpek started to give the pattern examples to teach generalization. For the first pattern (3,6,9…) which had triangle-shaped units (see Figure 4), she asked the students what the relationship could be among the terms of the pattern. Her specialized content knowledge to explain and justify a student's answer (SCK) appeared appropriately as in the script:

> İpek: How is the relationship for this pattern?
> Student: It goes 3 by 3.
> İpek: Yes, add 3 here (3 to 6), add 3 here (6 to 9). It goes like this. It increases by 3. What is the 4th term, the 5th term?
> Student: 12 and then 15.
> İpek: Okay, how can we find this relationship instead of counting one by one? Can we write a rule to find the 20th term? (She is writing '*a*' after the 20th term). For example, if what the 100th term is asked, do you write the terms to the 100th term?
> Student: I will multiply 100 by 3.
> İpek: Since it increases by 3, you say that you multiply 100 by 3. So, you multiply position number by increment.

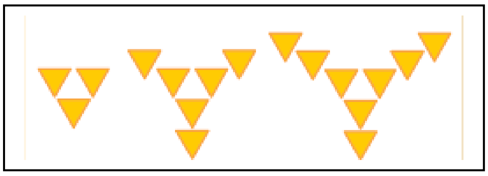

FIGURE 4. *The triangle-shaped units' pattern (Van de Walle,* Karp, & Bay-Williams*, 2013, p. 269)*

For the next questions, İpek wanted the students to find out the relationship in the pattern by multiplying the position number and the increment too. Besides, the teacher always wrote '*a*' as a general term after the 20th term as in the Figure 5. Thus, the students could think that the general term always comes after the 20th term, and they might also consider that it is the 21st term. This situation may prevent an understanding of the function of the variable, and thus, the general term. At this point, İpek could have stated that the 20th term is representative, or she could have used different term numbers in each question. She also should have emphasized that "*a*" was a variable or a general term, and it was substituted with any term number to find the term value.

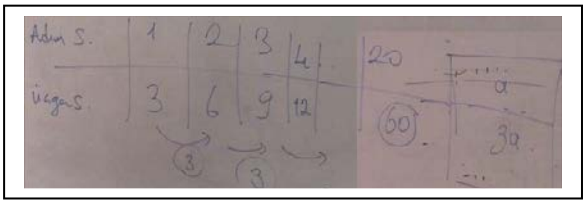

FIGURE 5. *The tabular representation of the pattern 3,6,9,12… used by İpek*

Class discussions

Throughout the instruction, the teacher sometimes made class discussions. In one of them, İpek asked the students to generalize the pattern 3,5,7,9... formed with matchsticks as in the below figure:

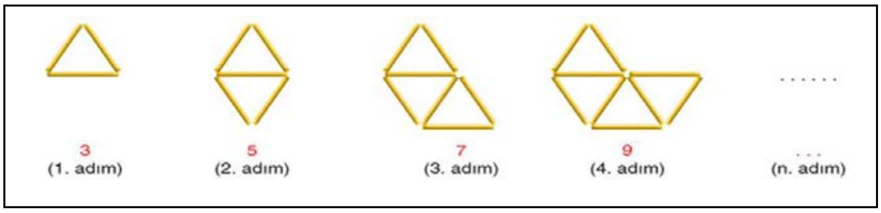

FIGURE 6. *The figural representation of the pattern 3,5,7,9… with matchsticks*

This pattern example was different from the first examples as the general rule of this pattern required to be written using a constant. In the discussion process, the teacher got the students' answers (2*n*, 3*n* and 5*n*), tried them to see if they worked and guided the students to get the general rule. In this generalization process, it is important to note how İpek used the representations (figural, tabular, and numerical) that she indicated in planning since it shows how her specialized content knowledge appeared (SCK). She focused on the arithmetic relationships by using a tabular representation to underlie the relationship in the pattern appropriately. However, she used the figures (matchsticks) merely to provide visuals. She should have emphasized the change of the figures and the relationship among the figures to guide the students to explore the relationship. For this pattern, she could have emphasized that the first figure was a triangle and had 3 matchsticks, and then, 2 matchsticks were added in each step, and one triangle was formed in each figure. She could have explained as the first figure had 1 matchstick at the bottom (1 matchstick was constant for each term), and then, 2 matchsticks were added in each step to form a triangle (2*n*). Thus, her specialized content knowledge to link figural and numerical representations underlying the idea of the relationship of the pattern appeared inadequate (SCK). Actually, she had emphasized using graphs and linking among the representations in lesson planning. However, she could not have explained the use of this representation in planning and she did not use it in the instruction, either. For the graphical representation, she could have drawn a graph using the position numbers and term values like in the Figure 7 and expected the students to find the equation (2*x*+1) of this graph. However, the students had not learned linear equations yet. Thus, İpek could neither show this representation in the lesson plan and instruction (SCK) nor did she have adequate knowledge about the students' prior knowledge (KCS).

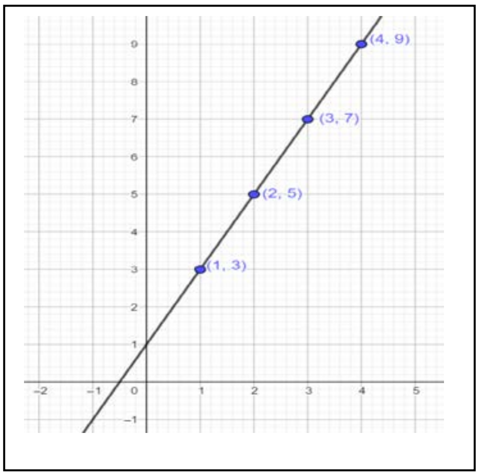

FIGURE 7. *The graphical representation of the pattern 3,5,7,9…*

Then, İpek wanted the students to generalize 2,6,12,20… as a non-linear figural growth pattern to make them focus on the topic deeper (KCT). The textbook (MoNE, 2014) had non-linear growth pattern examples and the teacher followed the textbook in the instruction process. İpek drew a table and wrote 'position numbers, terms, first representation and second representation' as headings in each column. To guide the students to generalize, she asked "the multiplication of the position number and the number of cubes" incorrectly to express the general rule as indicated below:

> İpek: How can we express the multiplication of the position number and the number of cubes? For example, when the position number is 1, here is 1.(1+1); when the position number is 2, it is 2.(2+1). When the position number is 3, it is 3.(3+1). When the position number is *n*, how can we write?

İpek's specialized content knowledge of how to provide a mathematical explanation for the general rule appeared inappropriate (SCK) since the number of cubes was the result and not the multipliers which formed the general rule. She should have emphasized that there was a relationship between the position number and the number of cubes. If she wanted to guide the students, she could have led the students to use the position number with operations or constants to get the number of cubes. Then, she could have explained numerical expressions (e.g. 1.(1+1), 2.(2+1)) as the multiplication of position number by 1 more of it. Then, she emphasized the second representation of the relationship:

> İpek: There is also the 2nd option here. We have learned the distributive property of multiplication on addition. The second option is about it. For example, we distribute 1, 1.1 + 1.1, what happens then, 1^2+1. Let's look at the second one, 2.2+2.1, how can we represent 2.2? How many times 2 is written?
> Students: 2.
> İpek: Then, 2 power of 2, 2^2+2. For the 3rd term, 3.3+3.1=3^2+3. If we write this representation using *n*, it is n^2+n.

She explained the second representation by showing the application of the distributive property on the first representation. Instead of using the distributive property, she could also have expanded the term values as 2=1+1, 6=4+2, 12=9+3, 20=16+4 … and then showed them as 2=1^2+1, 6=2^2+2, 12=3^2+3, 20=4^2+4… n^2+n in the table. Then, another student came up with another idea:

> Student: Teacher, I found the rule like this, I multiplied 1 with the next number, 1 and 2, then 2 and 3, 3 and 4. I could find the same results.
> İpek: How did you find the general rule?
> Student: I multiply the position number and the next position number.
> İpek: It is the same thing. Actually, we did like this, multiplying 1 and 2, 2 and 3, 3 and 4. That's good and correct.

The student found a relation between the position numbers as input values, but the relation was between the position number and the terms, input and output values. However, the teacher could not realize the student's incorrect reasoning and she accepted it as correct. After the lesson, in the post-observation interview, İpek asserted that the students had difficulty with the generalization of non-linear growth patterns and they did not understand it. She explained her impressions as in the following script:

> İpek: They did not understand the non-linear growth pattern in this lesson, the situation of multiplication of *n* by *n*. They did not make any efforts. They wanted to find the general rule immediately as in the linear growth pattern. So, the performances of the students were not good.

İpek suggested designing the lessons based on real-life situations in the post-interview and however she only explained that it required a well-structured design and took time to implement.

Explanations about generalization process

İpek's explanations for generalization of patterns were troublesome, and she could not anticipate possible misunderstandings while she was explaining. She generalized patterns such as finding the 20th term first and then writing *a* (as a variable) for the next column. This situation might cause the students to think that the general term came after the 20th term. İpek guided the students to multiply the difference and the position number, and to add a number to find the first term to get the general rule throughout the instruction. However, for the general rule of the pattern 4,8,12,16... 4*n*, she explained that adding or subtracting constants was not required for every generalization. This situation confused the students because this explanation contradicted her proposed strategy. Instead of it, she could have explained that the multiplication of the difference and position number yielded the first term, and there was no need to add a constant. This explanation could have been consistent with her proposed strategy. Another method that she proposed was adding the difference between the term and the position number, and the general term. To illustrate, in the pattern 3,4,5,6…, 2 from 3-1 (for the 1*st* term the position number=1 and the term=3), and she found the general rule as n+2 for this pattern. This situation caused such misconceptions that some students used this method for the next patterns and they generalized the pattern 4,6,8... as n+3. This situation also caused misconceptions in some students. These generalization ways of İpek could show her lack of knowledge of the content and students. Her knowledge of the students to anticipate the misunderstandings that might arise with pattern generalization methods being studied appeared inadequate (KCS). At least, İpek could have used the expression of "multiply the difference and the position number and add a number to find the first term" for preventing possible confusions and misconceptions. However, in the generalization process, the teacher was expected to emphasize looking for the relationship between the position number and the term. While generalizing patterns, İpek generally used tables. To illustrate, she drew a table with the first and second column like in Table 1. She could have also added the third column, included expanded term values that provided exploring the general rule of the pattern.

Table 1. *The tabular representation of the pattern (4,8,12,16…)*

The position numbers	The terms	The relation
1	4	4.1
2	8	4.2
3	12	4.3
4	16	4.4
….	…	…
n	…	…

The table allowed seeing the position numbers, and the teacher could have shown the relationship between the position numbers and the terms within the table easier than number patterns. She could have also modeled the number patterns with figures such as unit cubes or drawings on the board and then guided the students to think of figural change. In summary, the teacher could have used these representations in pattern generalization rather than using different formulas for each pattern.

The Case of Gül

Planning

To start the instruction, Gül chose the non-linear pattern activity (2,6,12,20…) from the textbook. She planned to use modeling to represent the pattern and to give a table to get the relationship of the pattern. Then, she planned to continue the instruction with linear growth patterns in the textbook such as 3,5,7,9… pattern. Her teaching knowledge to choose which examples to start with and which examples to use to take the students deeper into the content appeared with this sequence of the lesson plan inappropriately. In this sequence, it would be more appropriate, to begin with the linear growth pattern (3,5,7,9…) examples before the non-linear pattern generalization. Since this order might provide to teach pattern generalization from simple to complex ones developmentally and beginning with linear growth pattern generalization could also provide to remind students to pattern generalization from 6th grade as a prior concept. In generalizing patterns, she planned to use manipulative for modeling the pattern and then to use tables to underlie the relationship in pattern to make generalization easier for students. She explained the process of implementation of this activity (non-linear growth pattern) by finding several next terms in the table first and then discussing the generalization of the pattern by using the arithmetical representations in this table. She aimed to underlie the relationship of the pattern by using the tabular representation. She also planned to use the figural representation of patterns as merely using a different representation of the same pattern. She did not aim to use figural reasoning as of how the figures change and to relate this change to the relationship between numbers. Based on her explanations, her specialized content knowledge of how to use tabular representation was appropriate but her use of figural representation was not effective (SCK). However, she explained that the students would have difficulty in generalizing patterns algebraically and particularly in non-linear patterns. This shows her appropriate knowledge to anticipate the students' difficulty in generalizing patterns algebraically (KCS). At this point, Gül planned to use inductive reasoning to provide the students to think and use n for the nth term to get generalization. To do this, she planned to explain using the position number to find the corresponding terms such as 2 for the 2nd term, 3 for the 3rd term, 4 for the 4th term and then n for the nth term. This indicates her appropriate knowledge to build students' thinking with inductive reasoning for generalization of patterns (KCT).

Instruction and reflections

The findings of the instruction are presented under three headings as *beginning*, *class discussions*, and *explanations* to understand the teacher's knowledge of pattern generalization. The reflections are given where relevant to the situation.

Beginning

Gül's developed definition for the pattern concept with using students' answers appeared as "… there is an order, and it goes in a particular way. As you say, numbers or figures can be used. Then, we can say, the patterns are numbers or figures that continue based on a certain rule." In this situation, her specialized content knowledge to develop a usable definition appeared appropriate (SCK). Then, Gül taught patterns by centering the textbook and gave examples from the textbook as in its order without reminding linear-growth pattern generalization. She started with the first example (2,6,12,20…) of the textbook. This pattern was non-linear figural growth pattern and she modeled it as in Figure 8:

FIGURE 8. *The representation of the pattern (2,6,12,20…) with unit cubes*

Class discussions

In the generalization of the pattern 2,6,12,20…, Gül first modeled the pattern for the first four terms with unit cubes. Her specialized content knowledge on how to use mathematical representations appeared inadequate at this point (SCK) since she used figures to provide visuals for the students, and she had indicated it as a modeling in planning. This was because she did not use the change of figures to point out the growth to guide the students to explore the relationship. Actually, she used figural representations only to facilitate the students' imagination. The teacher should have used the features of figures for figural reasoning. For example, she could have used the area of the rectangle, multiplication of the length of the short side (n) by long side (n+1), that yields the number of cubes. Alternatively, she could have separated the figure as a square and a rectangle, the squared position number (n^2) and position number (n), since she did not use the figures. However, Gül used a table (see Figure 9) for the generalization of this pattern as for other patterns. She filled the table by asking students what the term is for each step. She could also have guided the students as explained in İpek's instruction process using the table.

In the discussion process, her teaching knowledge of when to use students' remarks to make a mathematical point appeared appropriate and the teacher asked how to find the 50th term by leading the students to find a general rule (KCT). She wanted the students to find a way different from counting one by one to the 50th term since she made the students feel that a rule for finding any terms in the pattern was needed. One student explored the rule as the multiplication of 1 by 2 is 2; multiplying 2 by 3 is 6; multiplying 3 by 4 is 12. When some students had difficulty in understanding some mathematical points, Gül paused for more clarification and made explanations about how they wrote the rule again:

Gül: You have found the 5^{th} term and understood how the other terms will go on. If the 20^{th}, 50^{th}, or 100^{th} term is asked, you cannot write to the 100^{th} term one by one. So, you should reach a generalization. We will find a relation between the position number and the number of cubes. Can we find it? Your friend said that he could find it by multiplying 1 by 2, multiplying 2 by 3, and 6; multiplying 3 by 4, and 12.

Gül explained the generalization process again and emphasized the student's solution, and wrote in the table as 1.2=2, 2.3=6, 3.4=12, 4.5=20, and 5.6=30 (see Figure 9). She always emphasized the relationship between the position number and the term. Her specialized content knowledge of how to provide mathematical explanations for functional thinking in patterns (SCK) appeared appropriate as in the script:

Gül: Okay, which of them mean what? For example, what 1 mean? Where do you take it?
Student: The position number
Gül: What 2 mean? Where do you take it?
Student: The number of cubes?
Gül: But, you will get the number of cubes. Let's call it anything else. Is it 1 more of the position number? (She is writing 1.(1+1), 2.(2+1), 3.(3+1), 4.(4+1), 5.(5+1)).

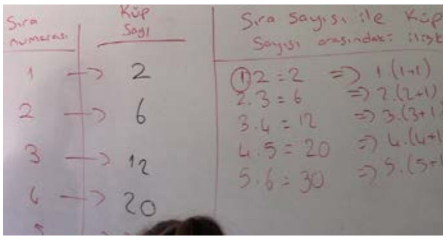

FIGURE 9. *The tabular representation of the pattern (2,6,12,20…) by Gül*

She realized the lack of knowledge of students about writing verbal statements algebraically when she asked 'What is 1 more of n? How can we present it?' one student answered as n+1, and the teacher wrote n.(n+1) as the general rule. She reminded algebraic expressions by giving several examples such as 5 more of a number, 5 less of a number, and then she connected 1 more of it as n+1. Her knowledge of the students to understand writing the algebraic expression as the difficulty of the students with the generalization of patterns appeared appropriate (KCS), and she also indicated this lack of knowledge of the students in the post-interview as in the following script:

Gül: Students could continue the pattern easily. It is important to note that they realized getting to the 50^{th} term one by one was not possible much. They thought about how they could find a short way. One student could find the relation arithmetically immediately. But, they had difficulty in generalization. They should solve different questions. This difficulty stems from algebraic expressions. They were taught in the previous year, but they could not remember.

Then, Gül continued the instruction process by generalizing different linear growth patterns from the textbook and other resources.

Explanations about generalization process

Gül generally made explanations when she noticed the students' possible misconceptions. To illustrate, after the generalization of the pattern 2,6,12,20… as n.(n+1) in the first lesson, some students overgeneralized as they considered that this rule worked for other patterns. She had wanted the students to generalize 3,5,7… as a linear growth numerical pattern. She drew a table and wrote the position number and the corresponding numbers in each column. She filled the rows with the first four terms of the pattern and then asked what the 100th term was. Some students proposed multiplying 100 by 101 to find the 100th term. At that point, she stated that this rule worked for the pattern 2,6,12,20… and it might not work for this pattern (3,5,7…) and other patterns. She also showed the students that multiplying 1 by 2 did not yield 3 as the 1st term. Another student proposed a different way and asked "When 1 is added with 2, it is 3. If we add 100 and 101, will it be correct?". In this situation, the student proposed adding the position number with the next position number to get the corresponding term and Gül noticed the student's error. Her teaching knowledge of how to address the students' errors appeared effective (KCT). She stated that the relationship was investigated between the position number and the term and not between the terms to prevent this misunderstanding of the student.

Although Gül stated that the relationship was analyzed between the position number and the term appropriately, not between the terms in the pattern, she also showed and asked the students to use a shortcut that was "multiply the difference between the terms and *n*, then add or subtract a number to find the first term if the difference is constant" throughout the instruction process. For example, her explanations for the pattern 5,8,11,14… whose general rule was $3n+2$ were as in the following script:

> B: It goes increasing by 3. So the rule begins with '3*n*'. Then by, the first term is 5. To get 5, when 1 is substituted for *n*, and 1 is multiplied by 3, that is 3. To get 5, what did you add? 2. You can find the general rule by using this way. Start with 3*n* and add 2. The general rule is $3n+2$. We can use this way for the patterns whose terms increase constantly.

This method is based on recursive strategy and appropriate in teaching pattern generalization. However, the students could perceive this phrase as a rule for generalization, and they might memorize it. This could prevent the students from conceptualizing the generalization process. Additionally, using only the first term to complete the general rule might cause misunderstandings, the students can consider the first term only while generalizing, and it may prevent understanding that the rule is valid for all the terms of the pattern. Her knowledge of the students to anticipate the misunderstandings that might arise with pattern generalization being studied in the class appeared inadequate (KCS). Thus, she could have made more explanations for the meaning and reasoning of the procedure and the general rule conceptually.

DISCUSSION and CONCLUSIONS

The purpose of this study is to analyze what forms of knowledge that mathematics teachers have, how they use them in their practices, and to propose what mathematical knowledge they need for teaching pattern generalization.

Based on the teachers' SCK, they used the tabular representation by focusing on the arithmetical relationships in the rows of the table to underlie the relationship in the pattern to conceptualize generalization effectively. As Warren and Cooper (2008) noted that finding the relationship between the position number and the term in the rows of the tables can provide the use of the tabular representation of patterns effectively, rather than only writing the numbers at the row and seeking the difference between the previous one. However, the teachers in this study used the figures or manipulative merely to provide visuals and they did not make any explanations about the change of the figures and the relationship between the numbers. İpek also had stated to use graphical representation to show the change of the terms in planning, but she could not explain the use of this representation for the pattern and she could not use it in the instruction. Akyüz, Coşkun, and Hacıömeroğlu (2009) concluded that transforming the relationship of the pattern between tabular, graphical, and numerical representations improved students' understanding of generalization.

Many studies suggested using figural reasoning to develop students' understanding of the relationship with considering the changes between the figures (e.g. Barbosa & Vale, 2015; Becker & Rivera 2005; Walkowiak, 2014). Actually, before transforming the number of figures to the table, the use of figural reasoning as asking the students how the units in the figures get together and what the relationship is based on provides to get the rule of the pattern conceptually and correctly (Thornton, 2001). For this study, the teachers used tabular and figural representations, but they always used numerical reasoning similar to pre-service teachers' use of recursive strategy for the general rule of the numerical pattern in Zazkis and Liljedahl's (2002) study. In lesson planning, the teachers had decided to use figural patterns when they noticed students' answers in the pattern test. However, they used the figures to provide merely visuals for the students instead of using figural reasoning as of how the figures change and relating this change to the relationship between terms. The teachers may have a lack of knowledge about figural reasoning and thus they could not use it. The studies show that the teachers did not have adequate knowledge to explain the strategies that students used for generalization of patterns (Baş et al., 2011; Callejo & Zapatera, 2017; El Mouhayar & Jurdak, 2013). Another decision of the teachers in planning was about using a table to show the relationship between input and output values of patterns. Gül always used tabular representation and continued to use it in this instruction. Ipek also used tables and expressed that the ease of exploring the relationship in patterns by using tables. The tabular representation was used to allow students to express their opinions and reasoning strategies from a different perspective (Warren, 1996).

Besides the use of representations, explanations for functional thinking are important in teaching. Smith (2008) explained functional thinking as "representational thinking that focuses on the relationship between two (or more) varying quantities" (p. 143). Generalizing patterns required functional thinking between the input and output values (Moss et al., 2008; Rivera, 2010). The teachers' explanations of the relationship in the patterns sometimes appeared differently. İpek explained that she had investigated the relationship between the output values or input values. This situation showed her lack of content knowledge about functional thinking in generalization. Consistent with this finding, Kutluk (2011) found that teachers seek the relationship between input or output values, and it showed inadequate content knowledge. The reason for this lack of knowledge could be that the teacher did not have a relational understanding of the functional relationship (Wilkie, 2014). On the other hand, Gül always emphasized that the relationship between the position number and the terms appropriately for functional thinking, and she also explained the function of general rule by showing the same arithmetical procedure worked for all the terms.

Based on the teachers' KCS, the teachers' anticipation of the difficulties that the students had in generalizing pattern algebraically and particularly non-linear patterns appeared clearly in the instructions. The students in this study they could extend the pattern as near generalization, but they had difficulty with getting the general rule algebraically as far generalization in linear growth patterns and mostly non-linear growth patterns (Ebersbach & Wilkening, 2007; Jurdak & El Mouhayar, 2014) Besides, İpek's explanations for generalization of patterns were troublesome as indicated findings section and they caused misconceptions in the learning of some students. As indicated in the studies (Baş et al., 2011; Callejo & Zapatera, 2017; El Mouhayar & Jurdak, 2013), İpek's inadequate knowledge of students' thinking could be derived from her strategies that showed her lack of content knowledge. On the other hand, Gül was seeking the relationship between input and output values in the pattern appropriately, she showed and asked the students to use the shortcut that was multiplying the difference between the terms and *n*, then to add or subtract a number to find the first term if the difference was constant throughout the instruction. This situation could cause the students to perceive it as a rule and also to consider the first term only while generalizing it and this could prevent that understanding the rule was valid for all the terms of the pattern as Wilkie (2014) reminded us. In sum, although their explanations were different at some points, the teachers' generalization methods were similar, they both extended the pattern based on the difference between the terms and found an algebraic rule considering the first term. Last, the two teachers stated that examining students' generalization strategies in planning increased them an awareness of students' possible misconceptions during the instruction. Studies have shown that knowing and analyzing student thinking helps teachers to provide more effective learning environments for their students and thus contribute to the success of students. (e.g. Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fennema et al., 1996).

Based on the teachers' KCT, İpek's teaching knowledge to design the tasks in terms of learning appeared appropriate. On the other hand, Gül's teaching knowledge to choose examples and activities to start with non-linear patterns and then to continue with the linear growth patterns to take students deeper into the pattern generalization was not appropriate developmentally. The teachers used the examples and activities from the textbook and they did not create the activities for the instructions. As Wilkie (2014) indicated, the teachers did not have adequate experiences and they could not create activities about pattern generalization to give the idea of functional thinking such as function machine.

While teaching pattern generalization, the two teachers used the recursive strategy that was focusing on the difference among consecutive output values in the pattern, and then, they used the explicit rule that was finding an algebraic rule relating input and output values in the pattern. These strategies are complementary and students are expected to transform the recursive strategy to the explicit rule (Healy & Hoyles, 1999; Lannin et al., 2006). The two teachers used them in generalizing number patterns and also figural patterns. In both number patterns and figural patterns, numerical reasoning may be used to get generalization. Numerical reasoning supports thinking of the relationship among term values and the recursive strategy based on it (Walkowiak, 2014). It is important to note that Gül guided the students to get explicit rules by using her suggested shortcut strategy throughout the instruction process. It was multiplying the difference between the terms and *n*, then adding or subtracting a constant to find the first term. The students used this strategy but whether they conceptualized generalization or not were questionable. On the other hand, although İpek mostly used this strategy, she sometimes made different and problematic explanations that were in contradiction with the shortcut as explained above. Gül also taught generalization inductively. She started specific examples, and then, she reached the generalization with the inductive reasoning appropriately. She used it to make the students feel the need for a general rule in the generalization of patterns. In the instruction processes, from the aspect of the students, Gül also addressed and remedied the students' errors in the generalization process by stating the relationship which was investigated between the position number and the term. Wilkie (2014) described KCT item for generalization of patterns by addressing a student's mistake in generalization and identifying appropriate strategies for conceptualizing functional thinking, also Gül addressed the errors and explained the functional relationship in the pattern appropriately. Another extracted knowledge as KCT is related to leading the discussion and Gül's knowledge generally emerged appropriately and adequately. Her conceptual knowledge provided the teacher with leading the discussion effectively. Similarly, Newton and Newton (2001) showed that teachers who had inadequate content knowledge could especially ask few questions to reveal the reasoning of ideas. In the learning perspective, Brown and Smith (1997) noted that making the students think by using questioning in classroom practices supported the students' understanding of algebra.

Summary

In sum, Table 2 shows the extracted knowledge types as a whole. This table is arranged based on İpek and Gül's planning and instructions, and literature (Baş et al., 2011; Callejo & Zapatera, 2017; El Mouhayar & Jurdak, 2013; Walkowiak, 2014; Wilkie, 2014). The descriptions include what mathematics teachers should know and how they should use their knowledge appropriately in teaching pattern generalization.

In general, the two teachers had the same types of knowledge in planning, but their practices in the instructions were not effective in the same manner because the teachers need to have conceptual subject matter knowledge to put effective practices (Forzani, 2014). Teachers should have a good conceptual mathematical understanding and also knowledge of students' understanding and needs to design their lessons with effective tasks (Anderson, White, & Sullivan, 2005; Hiebert, 1997). The appropriate and adequate knowledge has an impact on the teachers' practices to be effective and thus teachers' knowledge should be transferred for effective teaching to develop teacher practice (Doerr, 2004; Fennema & Franke, 1992; Ng, 2011). In the current study, the teachers need to have conceptual SCK. As Adoniou (2015) showed, teachers had a lack of knowledge about how to teach in practice and reasoning of content as SCK. Specialized knowledge is required for mathematics teachers to develop conceptual teaching practices (Hordern, 2015). When they have this knowledge, their knowledge of content and students (KCS) and knowledge of content and teaching (KCT) can also develop.

Table 2. *The extracted knowledge types for teaching pattern generalization*

Knowledge types	Description (specific to pattern generalization)
SCK	The knowledge to link and use representations (figural, numerical, and tabular) to underlying the functional relationship The knowledge to develop or choose a usable definition of the pattern, general term, and general rule concepts The knowledge of how to explain and justify the students' proposed general rules with substituting the position numbers in the general rule The knowledge of how to provide mathematical explanations for functional thinking in patterns
KCS	The knowledge to predict using figural patterns that the students find interesting and motivating The knowledge to anticipate the misunderstandings that might arise with pattern generalization methods being studied The knowledge to anticipate the students' difficulty in generalizing linear and non-linear growth pattern algebraically The knowledge to understand writing algebraic expression as the difficulty of the students with generalization of patterns The knowledge to understand using figural and numerical reasoning as the needs of students with pattern generalization
KCT	The knowledge to choose to start with simple linear growth patterns and to take students deeper into pattern generalization with non-linear growth patterns The knowledge to build students' thinking with the inductive reasoning for pattern generalization The knowledge to decide when to ask *n*th term to further learning of students in classroom discussion The knowledge to decide when the students have difficulty in the understanding pattern generalization in classroom discussion The knowledge of how to address students' errors in generalization and remedy them with explaining the functional thinking The knowledge of how to use recursive strategy and explicit rule in generalizing patterns
KCC	The knowledge to know the content and objectives in the curriculum related to generalization of patterns The knowledge to sequence, present the instruction developmentally with recalling students' prior knowledge and emphasize what was learned in previous grades

One of the important implications of this study is to propose knowledge types specific to pattern generalization. The knowledge types that mathematics teachers should have for teaching pattern generalization are in the table 2. The proposed knowledge types can be used to develop algebra knowledge models and thus this study might contribute to mathematics education literature, and particularly teacher knowledge of algebra. Additionally, teacher knowledge can be examined in other algebra topics such as equations, slope, and linear equations in middle grades to give a general overview of algebra knowledge of teachers. The proposed knowledge types can also be used to train pre-service teachers and in-service teachers. Mathematics teacher educators can take into consideration the importance of teacher knowledge for developing teachers and their teaching practice, as the finding of the current study, to improve the teacher training programs. These knowledge types can also inform the curriculum developers that the objectives may be detailed and developed concerning the expectations of the teachers as knowledge types. The definitions and descriptions of knowledge types for teaching pattern generalization from the current study also may shed light on forming the new objectives for the curriculum.

REFERENCES

Adoniou, M. (2015). Teacher knowledge: A complex tapestry. *Asia-Pacific Journal of Teacher Education*, *43*(2), 99-116.

Akyüz, D., Coşkun, Ş., & Hacıömeroğlu, E. S. (2009). An investigation into two preservice teachers' use of different representations in solving a pattern task. In S. L. Swars, D. W. Stinson, & S. Lemon-Smiths (Eds.), *Proceedings of the 31st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1261-1265). Atlanta, GA: Georgia State University.

Anderson, J., White, P., & Sullivan, P. (2005). Using a schematic model to represent influences on, and relationships between, teachers' problem-solving beliefs and practices. *Mathematics Education Research Journal, 17*(2), 9-38.

Arcavi, A., Kessel, C., Meira, L., & Smith, J. P. (1998). Teaching mathematical problem solving: An analysis of an emergent classroom community. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), *Research in collegiate mathematics education III* (vol. 7, pp. 1-70). USA: American Mathematical Society.

Asquith, P., Stephens, A. C., Knuth, E. J., & Alibali, M. W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable*.* *Mathematical Thinking and Learning: An International Journal, 9*(3), 249–272.

Ayalon, M., & Even, R. (2015). Students' opportunities to engage in transformational algebraic activity in different beginning algebra topics and classes. *International Journal of Science and Mathematics Education, 13*(2), 285-307.

Ball, D. L. (2000). Bridging practices: intertwining content and pedagogy in teaching and learning to teach. *Journal of Teacher Education, 51*(3), 241-247.

Ball, D. L., & Bass, H. (2002). Toward a practice-based theory of mathematical knowledge for teaching. In E. Simmt, & B., Davis (Eds.). *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group / Groupe Canadien d'Etude en Didactique des Mathématiques* (pp. 3-14). Edmonton, AB: CMESG/GCEDM.

Ball, D.L., & Forzani, F. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, *60*(5), 497–511.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389- 407.

Barbosa, A., & Vale, I. (2015). Visualization in pattern generalization: Potential and challenges. *Journal of the European Teacher Education Network, 10*, 57-70.

Baş, S., Çetinkaya, B., & Erbaş, A. K. (2011). Teachers' knowledge about ninth grade students' ways of algebraic thinking. *Education and Science, 36*(159), 41-55.

Becker, J. R., & Rivera, F. (2005). Generalization schemes in algebra of beginning high school students. In H. Chick, & J. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for Psychology of Mathematics Education* (vol. 4, pp. 121–128). Melbourne, Australia: University of Melbourne.

Bishop, J. W., & Stump, S. L. (2000). Preparing to teach in the new millennium: Algebra through the eyes of pre-service elementary and middle school teachers. In M. Fernandez (Ed.), *Proceedings of 22nd North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 107-113). Tucson, AZ: ERIC.

Blanton, M., & Kaput, J. (2001). Algebrafying the elementary mathematics experience. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.). *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra* (vol. 1, pp. 344-352). Australia: University of Melbourne.

Blanton, M. L., & Kaput, J. J. (2003). Developing elementary teachers: Algebra eyes and ears. *Teaching children mathematics, 10,* 70-77.

Brown, C. A., & Smith, M. S. (1997). Supporting the development of mathematical pedagogy. *The Mathematics Teacher, 90*(2), 138.

Cai, J., Ng S. F., & Moyer, J. C. (2011). Developing students' algebraic thinking in earlier grades: Lessons from China and Singapore. In J. Cai, & E., Knuth (Eds.). *Early algebraization: A global dialogue from multiple perspectives.* (pp. 25-41). Verlag Berlin Heidelberg: Springer.

Callejo, M. L., & Zapatera, A. (2017). Prospective primary teachers' noticing of students' understanding of pattern generalization. *Journal of Mathematics Teacher Education*, *20*(4), 309-333.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*(4), 499-531.

Chapman, O. (2006). Classroom practices for context of mathematics word problems. *Educational Studies in Mathematics, 62*(2), 211-230.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five traditions* (2nd ed.). Thousand Oaks, CA: Sage Publications.

Doerr, H. M. (2004). Teachers' knowledge and the teaching of algebra. In K. Stacey & H. Chick (Eds.), *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study* (pp. 267-290). Dordrecht, The Netherlands: Kluwer.
Ebersbach, M., & Wilkening, F. (2007). Children's intuitive mathematics: The development of knowledge about nonlinear growth. *Child Development, 78*(1), 296-308.
El Mouhayar, R. R., & Jurdak, M. E. (2013). Teachers' ability to identify and explain students' actions in near and far figural pattern generalization tasks. *Educational Studies in Mathematics, 82*(3), 379-396.
El Mouhayar, R. R., & Jurdak, M. E., (2016). Variation of student numerical and figural reasoning approaches by pattern generalization type, strategy use, and grade level. *International Journal of Mathematical Education in Science and Technology. 47*(2), 197-215.
English, L., & Warren, E. (1998). Introducing the variable through pattern exploration. *Mathematics Teacher, 91*(2), 166–170.
Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education*, 403-434.
Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.
Forzani, F. M. (2014). Understanding "core practices" and "practice-based" teacher education: Learning from the past. *Journal of Teacher Education*, *65*(4), 357-368.
Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.) *Second handbook of research on mathematics teaching and learning* (vol. 1, pp. 225-256). USA: Information Age Publishing.
Girit, D. & Akyüz, D. (2016). Algebraic thinking in middle school students at different grades: Conceptions about generalization of patterns. *Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education, 10*(2), 243-272.
Hargreaves, M., Threlfall, J., Frobisher, L. & Shorrocks-Taylor, D. (1999). Children's strategies with linear and quadratic sequences. In A. Orton (Eds.), *Pattern in the Teaching and Learning of Mathematics*. London: Cassell.
Healy, L., & Hoyles, C. (1999). Visual and symbolic reasoning in mathematics: Making connections with computers. *Mathematical Thinking and Learning, 1*, 59–84.
Hiebert, J. (1997). *Making sense: Teaching and learning mathematics with understanding.* Heinemann, Portsmouth, NH: Heineman.
Hill, H. C., Ball, D. B., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Teacher Education, 39*(4), 372-400.
Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts. In F. K. Lester (Ed.) *Second handbook of research on mathematics teaching and learning* (vol. 1, pp. 111-155). USA: Information Age Publishing.
Hollins, E. R. (2011). Teacher preparation for quality teaching. *Journal of Teacher Education*, *62*(4), 395-407.
Hordern, J. (2015). Teaching, teacher formation, and specialized professional practice. *European Journal of Teacher Education*, *38*(4), 431-444.
Jurdak, M. E., & El Mouhayar, R. R. (2014). Trends in the development of student level of reasoning in pattern generalization tasks across grade level. *Educational Studies in Mathematics, 85*(1), 75-92.
Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*(3), 223-252.
Kanes, C., & Nisbet, S. (1996). Mathematics-teachers' knowledge bases: Implications for teacher education. *Asia-Pacific Journal of Teacher Education*, *24*(2), 159-171.
Kutluk, B. (2011). *The investigation of elementary mathematics teachers' knowledge of students' difficulties related to pattern concept,* Unpublished Doctoral Dissertation, Dokuz Eylül University, Turkey.
Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006). Recursive and explicit rules: How can we build student algebraic understanding? *Journal of Mathematical Behavior, 25*, 299–317.
Magiera, M. T., van den Kieboom, L. A., & Moyer, J. C. (2013). An exploratory study of pre-service middle school teachers' knowledge of algebraic thinking, *Educational Studies in Mathematics, 84*, 93–113.
Malara, N. A., & Navarra, G. (2009). The analysis of classroom-based processes as a key task in teacher training for the approach to early algebra. In B. Clarke, B. Grevholm, & R. Millman (Eds.), *Tasks in Primary Mathematics Teacher Education* (pp. 235–262). Berlin: Springer.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation: Revised and expanded from qualitative research and case study applications in education*. San Francisco: Jossey-Bass.

Ministry of National Education (2014). Devlet Kitapları - 1525 (3rd ed.). MEB Yayınları - 5700

Moss, J., Beatty, R., Barkin, S., & Shillolo, G. (2008) "What is your theory? what is your rule?" fourth graders build an understanding of functions through patterns and generalizing problems. In C. Greenes, & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics: 70th NCTM Yearbook* (pp. 155–168). Reston, VA: National Council of Teachers of Mathematics.

Newton, D. P., & Newton, L. D. (2001). Subject content knowledge and teacher talk in the primary science classroom. *European Journal of Teacher Education*, *24*(3), 369-379.

Ng, D. (2011). Indonesian primary teachers' mathematical knowledge for teaching geometry: implications for educational policy and teacher preparation programs. *Asia-Pacific Journal of Teacher Education*, *39*(2), 151-164.

Rivera, F. D. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics, 73*(3), 297-328.

Saxe, G. B., Gearhart, M., & Seltzer, M. (1999). Relations between classroom practices and student learning in the domain of fractions. *Cognition and Instruction, 17,* 1–24.

Schoenfeld, A. H., Minstrell, J., & van Zee, E. (1999). The detailed analysis of an established teacher's non-traditional lesson. *The Journal of Mathematical Behavior, 18*(3), 281-325.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Silver, E. A., & Smith, T (1996). Building discourse communities in mathematics classrooms: A worthwhile but challenging journey. In P. C. Elliott (Ed.), *1996 Yearbook: communication in mathematics, K–12 and beyond* (pp. 20–28). Reston, VA: NCTM.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. L. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 133–160). New York: Taylor & Francis Group.

Stacey, K., & MacGregor, M. (2001). Curriculum reform and approaches to algebra. In R. Lins (Ed.), *Perspectives on school algebra* (pp. 141–153). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Steele, D. F., & Johanning, D. J. (2004). A schematic-theoretic view of problem solving and development of algebraic thinking. *Educational Studies in Mathematics, 57*(1), 65-90.

Thornton, S. J. (2001). New approaches to algebra: Have we missed the point? *Mathematics Teaching in the Middle School, 6*(7), 388.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. (2013). *Elementary and middle school mathematics teaching developmentally* (8th edition). United State of America: Pearson Education.

Walkowiak, T. A. (2014). Elementary and middle school students' analyses of pictorial growth patterns. *Journal of Mathematical Behavior, 33,* 56-71.

Warren, E. (1996). *Interaction between instructional approaches, students' reasoning processes, and their understanding of elementary algebra,* Unpublished dissertation, Queensland University of Technology, Australia.

Warren, E., & Cooper, T. (2008). Patterns that support early algebraic thinking in the elementary school. In C. Greenes, & R. Rubenstein (Eds.), *Algebra and Algebraic Thinking in School Mathematics: 70th NCTM Yearbook* (pp. 113–126). Reston, VA: National Council of Teachers of Mathematics.

Wilkie, K. J. (2014). Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *Journal of Mathematics Teacher Education*, *17*(5), 397-428.

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.). Thousand Oaks, California: Sage Publications.

Zazkis, R. & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49*, 379–402.